SIGNALS AND SYSTEMS
For
Graduate Aptitude Test in Engineering

By
Dr. N. Balaji
Professor of ECE
JNTUK

Session: 5

Topic : Fourier transform

Date : 15.05.2020

By

Dr. N. Balaji

Professor of ECE

Syllabus

Continuous-time signals: Fourier series and Fourier transform representations, sampling theorem and applications; Discrete-time signals: discrete-time Fourier transform (DTFT), DFT, FFT, Z-transform, interpolation of discrete-time signals;

LTI systems: definition and properties, causality, stability, impulse response, convolution, poles and zeros, parallel and cascade structure, frequency response, group delay, phase delay, digital filter design techniques.

Contents

Fourier Series – Observations and Limitations

Real world signals are rarely periodic.

Transient behaviour is common in Electronics and Communication Engineering

The discrete spectrum is sparse and cannot carry complex information

A different representation is needed for non-periodic signals.

Aperiodic Signal Representation in Frequency Domain

- A periodic continuous-time signal can be represented in frequency domain using Fourier series.

- But in general, signals are non periodic.

- To address this, we use Fourier transform

Fourier Transform

- Transformation is the process in which either a time domain signal is converted to frequency domain or frequency domain signal is Converted to time domain so that the signal analysis becomes easy.
- For any non-periodic signal as $T \to \infty \text{ implies } w_0 \to 0$
- The discrete spectrum of Fourier Series is converted to continuous spectrum in Fourier Transform.
- Extension of Fourier Series is Fourier Transform
- Fourier Transform is an extension of F.S to non-periodic signals.

Fourier Integral

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T}\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau}\, d\tau \right] e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T} \Rightarrow \frac{1}{T} = \frac{\omega_0}{2\pi}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau}\, d\tau \right] \omega_0 e^{jn\omega_0 t}$$

Let $\Delta\omega = \omega_0 = \dfrac{2\pi}{T}$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau}\, d\tau \right] e^{jn\omega_0 t} \Delta\omega$$

$$T \to \infty \Rightarrow d\omega = \Delta\omega \approx 0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_T(\tau) e^{-j\omega \tau}\, d\tau \right] e^{j\omega t}\, d\omega$$

Fourier Integral

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau\right]}_{F(j\omega)}e^{j\omega t}d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$ Synthesis

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$ Analysis

Fourier Series vs. Fourier Integral

Fourier Series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Period Function

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t}\, dt$$

Discrete Spectra

Fourier Integral:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t}\, d\omega$$

Non-Period Function

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

Continuous Spectra

Fourier Transform Pair

Inverse Fourier Transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

Synthesis

rad/sec

Fourier Transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Analysis

Fourier Transform

- Fourier Transform

$$x(t) \rightleftharpoons X(\omega)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$\omega = 2\pi f$$

$$d\omega = 2\pi df$$

I.F.T → $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$

F.T → $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f} \, dt$

$$2\pi\delta(\omega) = \delta(f)$$

Proof: $2\pi\delta(\omega) = 2\pi\delta(2\pi f)$

$$= \frac{2\pi}{|2\pi|}\delta(f)$$

$$\therefore 2\pi\delta(\omega) = \delta(f)$$

Example Problem

- If X(t) is a voltage waveform, then what are the units of $X(f)$

- **Sol**:

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2} \quad dt$$

= volts. sec

- So $X(f)$ unit is volts.sec or volts/Hz

Conditions for existence of Fourier Transform

- **Conditions for existence of F.T.: (Dirichlet's Conditions)**

1. Signal should have finite number of maxima & minima over finite interval.
2. Signal should have finite number of discontinuities over finite interval.
3. Signal should have absolutely integrable.

$$i.e. \int_{-\infty}^{\infty} |x(t)|dt < \infty \begin{array}{l} \rightarrow \text{ Impulse signal} \\ \rightarrow \text{ Energy Signal} \end{array}$$

- Dirichlet's conditions are sufficient but not necessary.

Calculate Fourier Transform for a given Signal

- **Q: Calculate Fourier Transform for the signal**

$$x(t) = e^{-at}u(t), a > 0$$

- **Sol:**

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} = \frac{e^{-(a+j\omega)\infty} - e^{0}}{-(a+j\omega)}$$

Solution

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} = \frac{e^{-(a+j\omega)\infty} - e^0}{-(a+j\omega)}$$

$$e^{-(a+j\omega)\infty} = e^{-a\infty} \cdot e^{-j\omega\infty}$$

$$e^{-a\infty} = 0, a > 0$$

$$= \frac{0-1}{-(a+j\omega)}$$

$$\therefore X(\omega) = \frac{1}{(a+j\omega)}$$

$$e^{-j\infty} = cos\infty - jsin\infty$$

- The cos & sin functions are not defined in the given range.

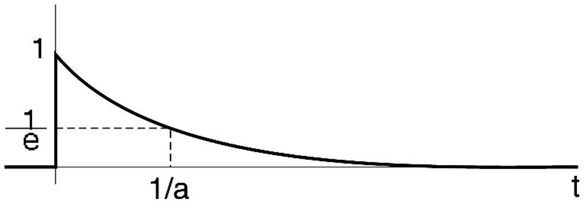
- At $t = \pm\infty$, complex exponentials & sinusoidal functions are undefined

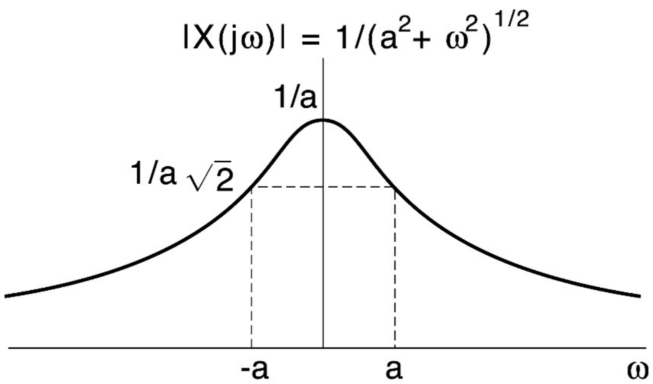
Fourier Transform of Right-Sided Exponential

$$x(t) = e^{-at}u(t), a > 0$$

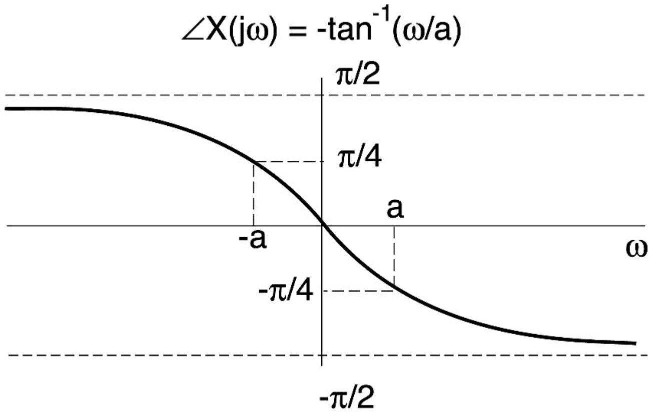
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)t}} dt$$

$$= -\left(\frac{1}{a+j\omega}\right) e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$





Even symmetry



Odd symmetry

Properties of Fourier Transform

$$Linearity \rightarrow a_1 x_1(t) + a_2 x_2(t) \rightleftharpoons a_1 X_1(w) + a_2 X_2(w)$$

$$Time\ reversal \rightarrow x(-t) \rightleftharpoons X(-w)$$

$$Conjuction \rightarrow x^*(t) \rightleftharpoons X^*(-w)$$

$$Time\ shif\,ting \rightarrow x\left(t - t_0\right) \rightleftharpoons X(w)\, e^{-jwt_0}$$

$$Time\ scaling \rightarrow x(at)\ (a \neq 0) \rightleftharpoons \frac{1}{|a|} X\left(\frac{w}{a}\right)$$

$$Freq.\ shif\,ting \rightarrow e^{-w_0 t} x(t) \rightleftharpoons X\left(w + w_0\right)$$

$$Dif\!f.\ in\ time \rightarrow \frac{d^n x(t)}{dt^n} \rightleftharpoons (jw)^n X(w)$$

Properties of Fourier Transform

$$\text{Integration in time} \rightarrow \int_{-\infty}^{t} x(t)\,dt \rightleftharpoons \frac{X(w)}{jw} + \pi X(0)\,.\delta(w)$$

$$\text{Convolution in time} \rightarrow x_1(t) * x_2(t) \rightleftharpoons \left[X_1(w) \cdot X_2(w)\right]$$

$$\text{Multiplication in time} \rightarrow x_1(t) \cdot x_2(t) \rightleftharpoons \frac{1}{2\pi}\left[X_1(w) * X_2(w)\right]$$

$$x_1(t) \cdot x_2(t) \rightleftharpoons \left[X_1(f) * X_2(f)\right]$$

$$Diff\,.\ in\ freq. \rightarrow t^n x(t) \rightleftharpoons (j)^n \frac{d^n X(w)}{dw^n}$$

$$\text{Parsval's energy theorem} \rightarrow E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(w)|^2 dw$$

Properties of Fourier Transform

$$Modulation \rightarrow x(t)\cos w_0 t \rightleftharpoons \frac{1}{2}\left[X\left(w+w_0\right)+X\left(w-w_0\right)\right]$$

$$x(t)\sin w_0 t \rightleftharpoons \frac{1}{2}\left[X\left(w+w_0\right)-X\left(w-w_0\right)\right]$$

$$Area\ of\ \ time-domain \rightarrow X(w)=\int_{-\infty}^{\infty} x(t)\,e^{-jwt}\,\mathrm{d}t$$

$$X(0)=\int_{-\infty}^{\infty} x(t)\,\mathrm{d}t$$

Properties of Fourier Transform

$$Area\ under\ Freq.\ Domain \rightarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\, {}^{jwt} dw$$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\, dw$$

$$\int_{-\infty}^{\infty} X(w)\, dw = 2\pi x(0)$$

$$\text{Area under } X(w) = 2\pi x(t)\ _{t=0}$$

Proof of Properties of Fourier Transform-Linearity

1. Linearity (Superposition)

If $x_1(t) \Leftrightarrow X_1(\omega)$ and $x_2(t) \Leftrightarrow X_2(\omega)$

Then, $a_1x_1(t)+a_2x_2(t) \Leftrightarrow a_1X_1(\omega)+a_2X_2(\omega)$

Proof:

$$\int_{-\infty}^{\infty}[a_1x_1(t)+a_2x_2(t)]e^{-j\omega t}dt = a_1\int_{-\infty}^{\infty}x_1(t)e^{-j\omega t}dt + a_2\int_{-\infty}^{\infty}x_2(t)e^{-j\omega t}dt$$

$$= a_1X_1(\omega)+a_2X_2(\omega)$$

Proof of Properties of Fourier Transform- Time Shifting

2. Time Shifting

If $x(t) \Leftrightarrow X(\omega)$

Then, $x(t - t_0) \Leftrightarrow X(\omega)e^{-j\omega t_0}$

Proof: Let $\tau = t - t_0$ then $t = \tau + t_0$ and $dt = d\tau$

$$\int_{-\infty}^{\infty} x(t - t_0)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau + t_0)}d\tau$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau$$

$$= e^{-j\omega t_0}X(\omega)$$

Proof of Properties of Fourier Transform- Time Scaling

3. Time Scaling

If $x(t) \Leftrightarrow X(\omega)$ then

$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Proof: Let $\tau = at$ then $t = \tau / a$ and $dt = (1/a)d\tau$

If , a>0 then

$$\int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\frac{\omega}{a}\tau}\frac{1}{a}d\tau$$

$$= \frac{1}{a}X\left(\frac{\omega}{a}\right)$$

Proof of Properties of Fourier Transform- Duality

4. Duality (Symmetry)

If $x(t) \Leftrightarrow X(\omega)$ then

$$X(t) \Leftrightarrow 2\pi\, x(-\omega)$$

or

$$X(t) \Leftrightarrow x(-f)$$

Proof: Since t and ω are arbitrary variables in the inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

we can replace ω with t and t with $-\omega$ to get

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt$$

Therefore,

$$F\{X(t)\} = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt = 2\pi x(-\omega)$$

Proof of Properties of Fourier Transform- Time Reversal

Time Reversal

If $x(t) \Leftrightarrow X(\omega)$ then

$$x(-t) \Leftrightarrow X(-\omega)$$

Proof:	Let $-t = \tau$. Then $t = -\tau$ and $dt = -d\tau$

$$\int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt = -\int_{-\infty}^{\infty} x(\tau)e^{-j(-\omega)\tau}d\tau = X(-\omega)$$

Proof of Properties of Fourier Transform- Frequency Shifting

Frequency Shifting

If $x(t) \Leftrightarrow X(\omega)$ then

$$x(t)e^{-j\omega_c t} \Leftrightarrow X(\omega - \omega_c)$$

Proof:

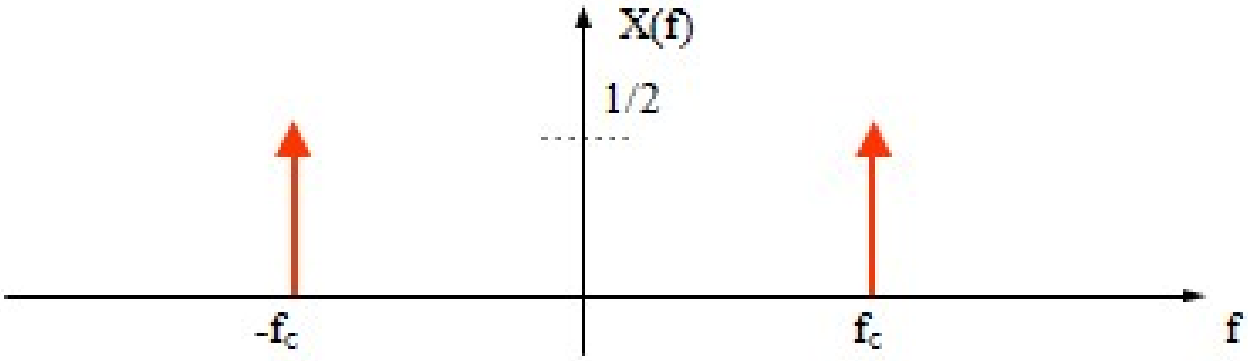
$$\int_{-\infty}^{\infty} x(t)e^{j\omega_c t}e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_c)t}\,dt = X(\omega - \omega_c)$$

Example: Determine the Fourier transform of $\cos\omega_c t$ and $\sin\omega_c t$

$$x(t) = \cos\omega_c t = \frac{1}{2}e^{j\omega_c t} + \frac{1}{2}e^{-j\omega_c t} \quad \Leftrightarrow \quad X(\omega) = \pi\left[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)\right]$$

or

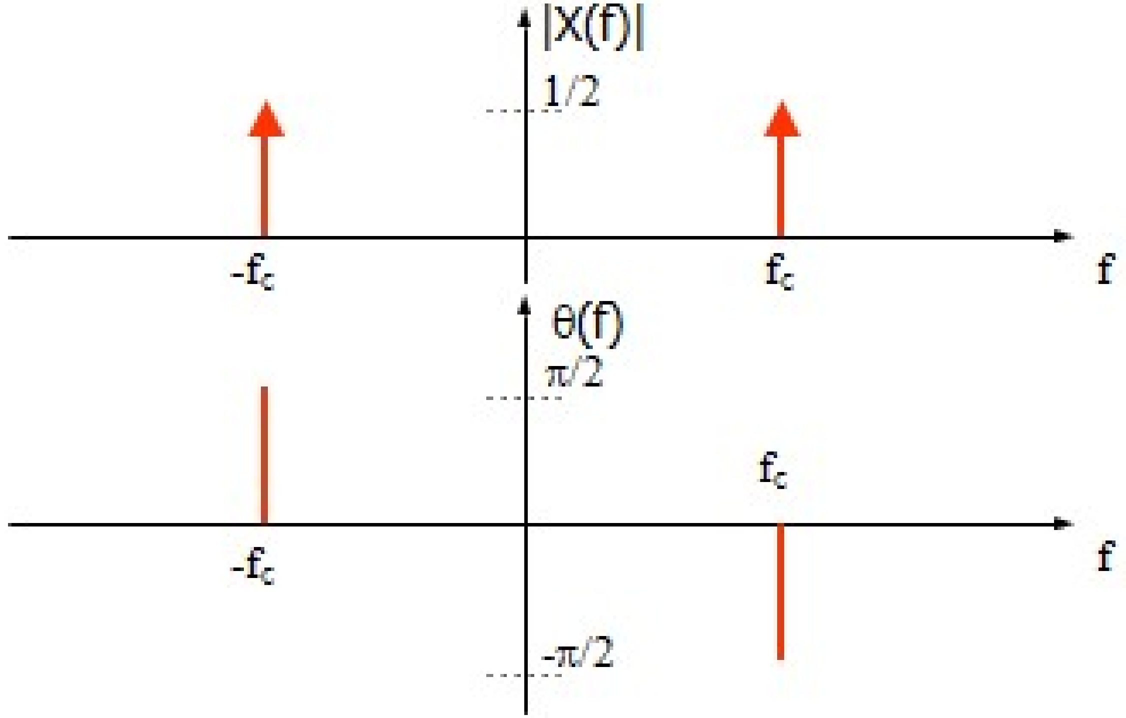
$$x(t) = \cos\omega_c t = \frac{1}{2}e^{j\omega_c t} + \frac{1}{2}e^{-j\omega_c t} \quad \Leftrightarrow \quad X(f) = \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$



The phase spectrum is zero everywhere.

$$x(t) = \sin\omega_c t = \frac{1}{2j}e^{j\omega_c t} - \frac{1}{2j}e^{-j\omega_c t} \Leftrightarrow X(\omega) = -j\pi\left[\delta(\omega-\omega_c) - \delta(\omega+\omega_c)\right]$$

$$x(t) = \sin\omega_c t = \frac{1}{2j}e^{j\omega_c t} - \frac{1}{2j}e^{-j\omega_c t} \Leftrightarrow X(f) = \frac{-j}{2}\left[\delta(f-f_c) - \delta(f+f_c)\right]$$



Proof of Properties of Fourier Transform- Modulation

7. Modulation

If $x(t) \Leftrightarrow X(\omega)$ then

$$x(t)\cos(\omega_c t) \Leftrightarrow \frac{1}{2}\left[X(\omega-\omega_c)+X(\omega+\omega_c)\right]$$

Proof:

$$\int_{-\infty}^{\infty} x(t)\cos(\omega_c t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)\frac{1}{2}\left[e^{j\omega_c t}+e^{j\omega_c t}\right]e^{-j\omega t}dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_c)t}dt + \int_{-\infty}^{\infty} x(t)e^{-j(\omega+\omega_c)t}dt\right]$$

$$= \frac{1}{2}\left[X(\omega-\omega_c)+X(\omega+\omega_c)\right]$$

Proof of Properties of Fourier Transform- Time Differentiation

8. Time Differentiation:

If $x(t) \Leftrightarrow X(\omega)$ then

$$\frac{dx(t)}{dt} \Leftrightarrow j\omega X(\omega)$$

General case

$$\frac{d^n x(t)}{dt^n} \Leftrightarrow (j\omega)^n X(\omega)$$

Proof: Taking the derivative of the inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

we obtain

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(\omega)e^{j\omega t}d\omega$$

Therefore

$$\frac{dx(t)}{dt} \Leftrightarrow j\omega X(\omega)$$

Proof of Properties of Fourier Transform- Convolution

11. Convolution

If $x(t) \Leftrightarrow X(\omega)$, $h(t) \Leftrightarrow H(\omega)$, and $y(t) \Leftrightarrow Y(\omega)$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$Y(\omega) = H(\omega)X(\omega)$$

Proof:

$$Y(\omega) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right]e^{-j\omega t}dt$$

Interchanging the order of integration, we obtain

$$Y(\omega) = \int_{-\infty}^{\infty} h(\tau)\left[\int_{-\infty}^{\infty} x(t-\tau)e^{-j\omega t}dt\right]d\tau$$

$$Y(\omega) = \int_{-\infty}^{\infty} h(\tau)X(\omega)e^{-j\omega\tau}d\tau = X(\omega)\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

$$= X(\omega)H(\omega)$$

Fourier Transform for Real Functions

If $f(t)$ is a real function, and $F(j\omega) = F_R(j\omega) + jF_I(j\omega)$

➡ $F(-j\omega) = F^*(j\omega)$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F^*(j\omega) = \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt = F(-j\omega)$$

Fourier Transform for Real Functions

If $f(t)$ is a real function, and $F(j\omega) = F_R(j\omega) + jF_I(j\omega)$

⇒ $F(-j\omega) = F^*(j\omega)$

⇒ $F_R(j\omega)$ is even, and $F_I(j\omega)$ is odd.

$F_R(-j\omega) = F_R(j\omega)$ $F_I(-j\omega) = -F_I(j\omega)$

⇒ *Magnitude spectrum* $|F(j\omega)|$ is even, and *phase spectrum* $\phi(\omega)$ is odd.

Fourier Transform for Real Functions

If $f(t)$ is real and even

$\Rightarrow$ $F(j\omega)$ is real ✓

Pf)

Even $\Rightarrow$ $f(t) = f(-t)$

$\Rightarrow$ $F(j\omega) = F(-j\omega)$

Real $\Rightarrow$ $F(-j\omega) = F^*(j\omega)$

$\Rightarrow$ $F(j\omega) = F^*(j\omega)$

If $f(t)$ is real and odd

$\Rightarrow$ $F(j\omega)$ is pure imaginary ✓

Pf)

Odd $\Rightarrow$ $f(t) = -f(-t)$

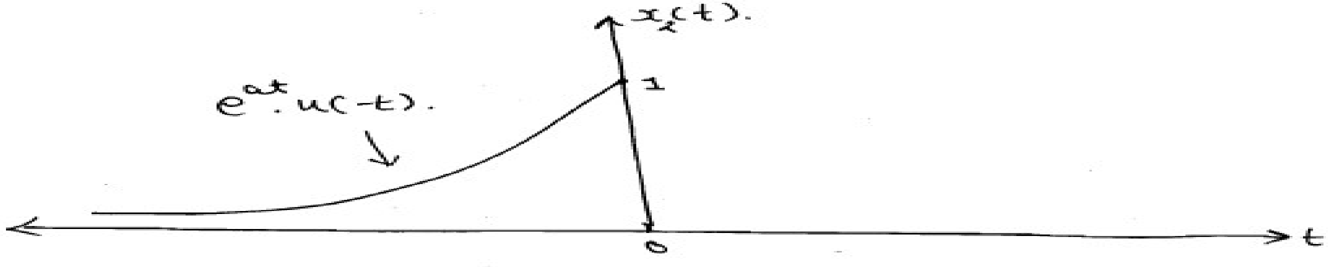
$\Rightarrow$ $F(j\omega) = -F(-j\omega)$

Real $\Rightarrow$ $F(-j\omega) = F^*(j\omega)$

$\Rightarrow$ $F(j\omega) = -F^*(j\omega)$

Example Problem -Fourier Transform of e^{at} u(-t) for a>0

2) e^{at} u(-t)   a>0   =>



Using time reversal property

$$F\big(x(t)\big) \overset{F.T}{\leftrightarrow} X(\boldsymbol{\omega})$$

$$F\big(x(-t)\big) \overset{F.T}{\leftrightarrow} X(-\boldsymbol{\omega})$$

$$X(-\boldsymbol{\omega}) = \frac{1}{a - j\boldsymbol{\omega}}$$

$$e^{at}u(-t) \overset{F.T}{\leftrightarrow} \frac{1}{a - j\boldsymbol{\omega}}$$

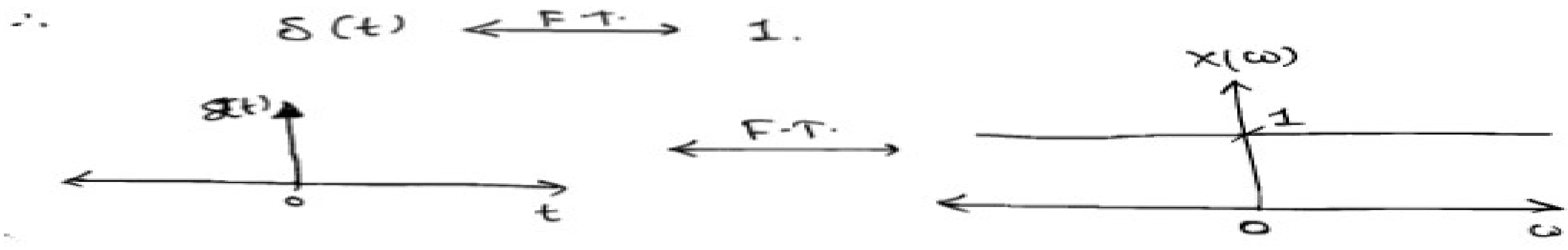
Example Problem -Fourier Transform of $\boldsymbol{\delta(t)}$

3) $x(t) = \delta(t)$

$$X(\boldsymbol{\omega}) = \int_{-\alpha}^{\alpha} \delta(t).\, e^{-i\omega t}.\, dt$$

$X(\boldsymbol{\omega}) = e^{-i\omega(0)}$ (since t=0, time shifting property of impulse)

$X(\boldsymbol{\omega}) = 1$



Spectrum of impulse is constant for the frequency

Example Problem -Fourier Transform of $e^{-a|t|}$ $a > 0$

Que: $y(t) = e^{-a|t|}$ $a > 0$ $find\ Y(\boldsymbol{\omega})$

Sol: $y(t) = e^{-a|t|}$

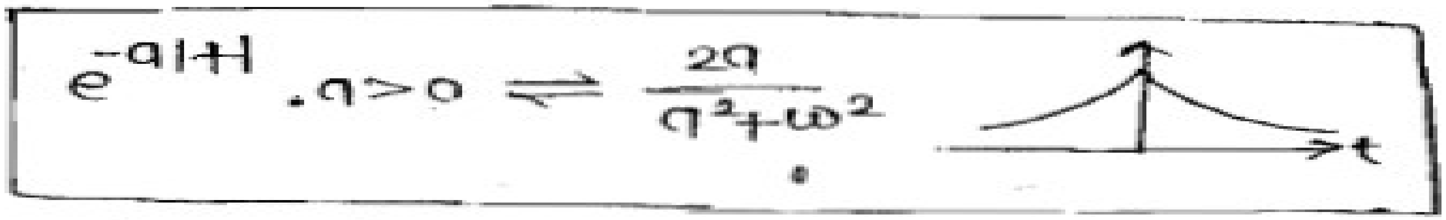
$= e^{at}$ t<0,

$= e^{-at}$ t>0,

$= e^{at}\,u(-t) + e^{-at}\,u(t)$

$$Y(\boldsymbol{\omega}) = \frac{1}{a - j\boldsymbol{\omega}} + \frac{1}{a + j\boldsymbol{\omega}}$$

$$Y(\boldsymbol{\omega}) = \frac{2a}{a^2 + \boldsymbol{\omega}^2}$$



Duality Property

- The **Duality Property** tells us that if x(t) has a **Fourier Transform** X(ω)
- If we form a new function of time that has the functional form of the **transform**, X(t), it will have a **Fourier Transform** x(ω) that has the functional form of the original time function (but is a function of frequency).

$$x(t) \leftrightarrow X(\omega)$$

$$X(t) \leftrightarrow 2\pi\, x(-\omega)$$

Duality Property

Property of duality

$$x(t) \rightleftharpoons X(w) \quad (t = -w)$$

$$X(t) \rightleftharpoons 2\pi x(-w)$$

$$x(t) \rightleftharpoons X(f) \quad (t = -f)$$

$$X(t) \rightleftharpoons x(-f)$$

Duality Property based Problem -Fourier Transform of $\frac{1}{a+jt}$

Q: $x(t) = \dfrac{1}{a + jt} \Longleftrightarrow x(\omega)=?$

Sol:

$$x(t) = \frac{1}{a + jt}$$

$(t = \omega)$

$$e^{-at}u(t),\ a>0 \Longleftrightarrow \frac{1}{a + j\omega}$$

$(\omega = t)$ $(t = -\omega)$

$$\frac{1}{a + jt} \Longleftrightarrow 2\pi\, e^{a\omega}u(-\omega),\ a>0$$

Duality Property based Problem -Fourier Transform of $\frac{2a}{a^2+t^2}$

Q: $x(t) = \frac{2a}{a^2+t^2} \Longleftrightarrow x(\omega)=?$

Sol:

$$x(t) = \frac{2a}{a^2+t^2}$$

$(t = \omega)$

$$e^{-a|t|},\ a>0 \Longleftrightarrow \frac{2a}{a^2+\omega^2}$$

$(\omega = t)$ $(t = -\omega)$

$$\frac{2a}{a^2+t^2} \Longleftrightarrow 2\pi\, e^{-a|-\omega|},\ a>0$$

$$\frac{2a}{a^2+t^2} \Longleftrightarrow 2\pi\, e^{-a|\omega|},\ a>0$$

Duality Property based Problem- Fourier Transform of A_0

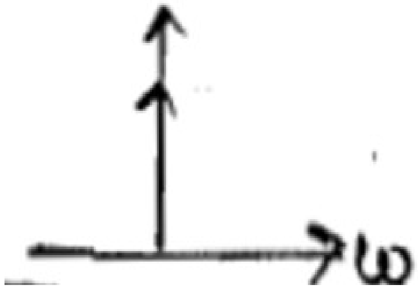
Q: $x(t) = A_0 \Longleftrightarrow$ x (ω)=?

Sol: $A_0\delta(t) \Longleftrightarrow A_0$

$(\omega = t)$ $(t = -\omega)$

$A_0 \Longleftrightarrow 2\pi A_0\,\delta(-\omega)$

A_0= DC Signal $\Longleftrightarrow 2\pi A_0\,\delta(\omega)$



Q:

Find the Y (ω) in terms of x (ω)

$x(t) \longleftrightarrow X(\omega)$

$y(t) \longleftrightarrow Y(\omega)$

(i) $y(t) = e^{j2t}x(t)$

Sol: $Y(\omega) = x(\omega - 2)$

(ii) $y(t) = x(-2t)$

Sol: $Y(\omega) = \frac{1}{2}x\left(\frac{-\omega}{2}\right)$

(iii) $y(t) = x(2t - 3)$

Sol: $y(t) = x(2t - 3) = x[2\left(t - \frac{3}{2}\right)]$

Scaling → $X_1(\omega) = \frac{1}{2}x\left(\frac{\omega}{2}\right)$

Shifting → $X_2(\omega) = x(\omega)\, e^{-j1.5\omega}$

$$Y(\omega) = \frac{1}{2}x\left(\frac{\omega}{2}\right)e^{-j\ .5\omega}$$

Frequency Shifting property

$e^{-j\omega_0 t}x(t) \longleftrightarrow x(\omega + \omega_0)$

Time Scaling property

$x(at),\ a \neq 0 \longleftrightarrow \frac{1}{|a|}x\left(\frac{\omega}{a}\right)$

Time Shifting property

$x(t - t_0) \longleftrightarrow e^{-j\omega t_0}\, x(\omega)$

(iv) $If\ y(t) = x(-2t-4)$ Find its Fourier Transform

Sol: $y(t) = x[-2(t+2)]$

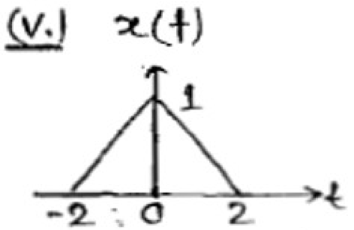
Scaling $a = -2$

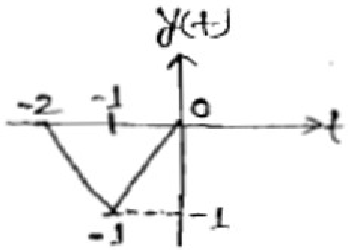
Shifting $t_0 = 2$

Time Scaling: $x(at),\ a \neq 0 \longleftrightarrow \frac{1}{|a|}\mathrm{x}\left(\frac{\omega}{a}\right)$

Time Shifting property: $x(t-t_0) \longleftrightarrow \mathrm{e}^{-j\omega t_0}\ \mathrm{x}\ (\omega)$

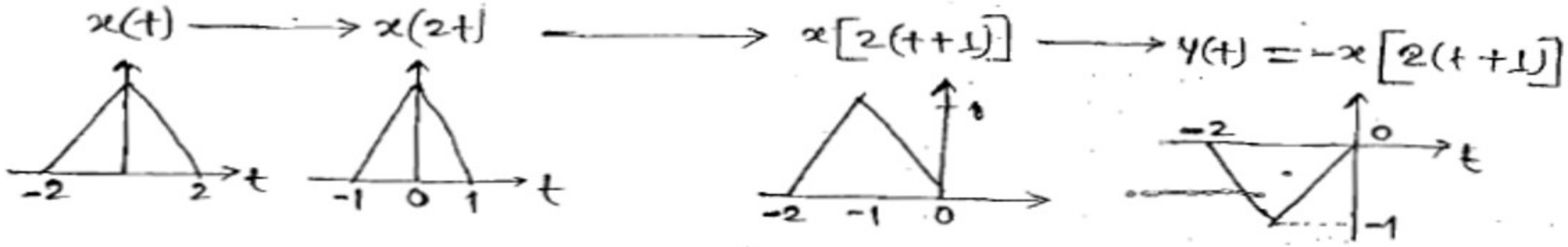
$$\mathrm{Y}(\omega) = \frac{1}{2}\mathrm{x}\left(\frac{-\omega}{2}\right)\mathrm{e}^{j2\omega}$$





$\mathrm{Y}(\omega) = ?$

Sol:



$$y(t) = -x[(2(t+1)]$$

$$\mathrm{Y}(\omega) = -\frac{1}{2}\mathrm{x}\left(\frac{\omega}{2}\right)\mathrm{e}^{j\omega} \quad where\ t_0 = 1$$

Q: $y(t) = x(t) * h(t) \longrightarrow$ (i)
$g(t) = x(3t) * h(3t) \longrightarrow$ (ii)

$if\ g(t) = Ay(Bt)$ then calculate A and B

Sol:

From equation (i)

$Y(\omega) = x(\omega)H(\omega) \longrightarrow$ (iii)

From equation (ii)

$$G(\omega) = \left[\frac{1}{3}x\left(\frac{\omega}{3}\right)\right]\left[\frac{1}{3}H\left(\frac{w}{3}\right)\right]$$

$$G(\omega) = \frac{1}{9}\left[x\left(\frac{\omega}{3}\right)H\left(\frac{w}{3}\right)\right]$$

$$G(\omega) = \frac{1}{9}\left[Y\left(\frac{\omega}{3}\right)\right]$$ From equation (iii)

$$= \frac{1}{3}\left[\frac{1}{3}\left(Y\left(\frac{\omega}{3}\right)\right)\right]$$

$$g(t) = \frac{1}{3}[y(3t)]$$ By comparing with $g(t) = Ay(Bt)$

$$A = \frac{1}{3} \qquad B = 3$$

Second method:

$$y(t) = x(t) * h(t)$$

$$x(at) * h(at) = \frac{1}{|a|}y(at)$$

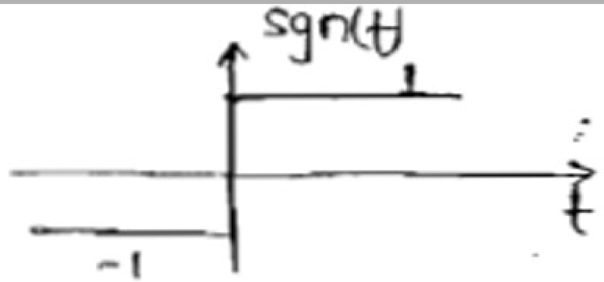
$$a = 3$$

$$x(3t) * h(3t) = \frac{1}{3}[y(3t)]$$

By comparing with $g(t) = Ay(Bt)$

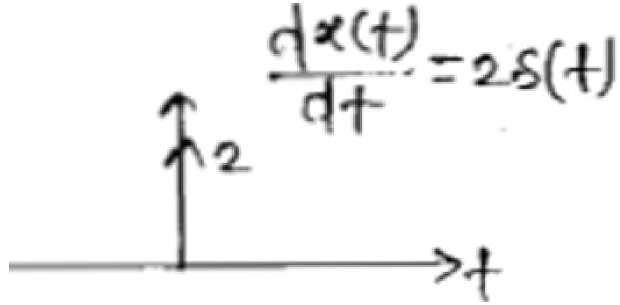
$$A = \frac{1}{3} \qquad B = 3$$

Q: $x(t) = \text{sgn}(t) \Longleftrightarrow X(\omega)$=?



$$\text{sgn}(t) = \begin{cases} 1 & , \; t > 0 \\ 0 & , \; t = 0 \\ -1 & , \; t < 0 \end{cases} = 2u(t) - 1$$

Sol:



$$x(t) = \text{sgn}(t) \Longleftrightarrow x(\omega) = \frac{2}{jw}$$

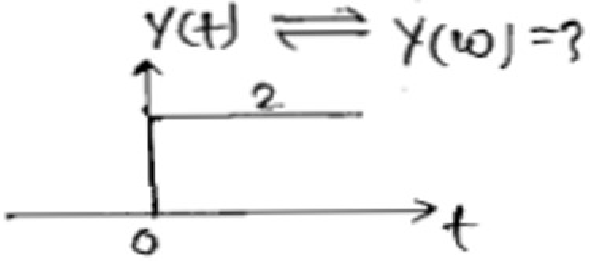
$$\frac{dx(t)}{dt} = 2\delta(t)$$

FT ↓

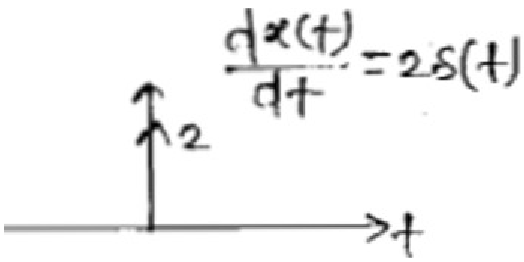
$$j\omega \, x(\omega) = 2$$

Find the Fourier Transform of the following Function

Q:



Sol:



$$avg = \frac{2}{2}$$

$$=1$$

$$= 2\pi\delta(\omega)$$

$$\frac{dy(t)}{dt} = 2\delta(t)$$

$$j\omega Y(\omega) = 2$$

$$Y(\omega) = \frac{2}{jw}$$ → wrong

Second method:

$$y(t) = 1 + x(t)$$

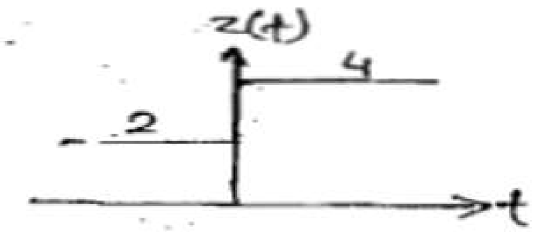
↓ FT

$$Y(\omega) = 2\pi\delta(\omega) + x(\omega)$$

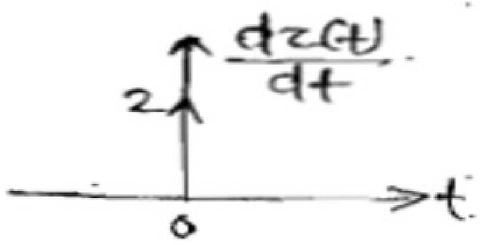
$$Y(\omega) = 2\pi\delta(\omega) + \frac{2}{jw}$$

Find the Fourier Transform of the following Function

Q:



Sol:



$$\frac{dz(t)}{dt} = 2\delta(t)$$

$$j\omega Z(\omega) = 2$$

$$Z(\omega) = \frac{2}{jw}$$ → wrong

$$avg = \frac{4+2}{2}$$

$$= 3$$

$$= 3 * 2\pi\delta(\omega) \quad = 6\pi\delta(\omega)$$

$$Z(\omega) = 6\pi\delta(\omega) + \frac{2}{jw}$$

Second method:

$$z(t) = 3 + x(t)$$

↓ FT

$$Z(\omega) = 6\pi\delta(\omega) + x(\omega)$$

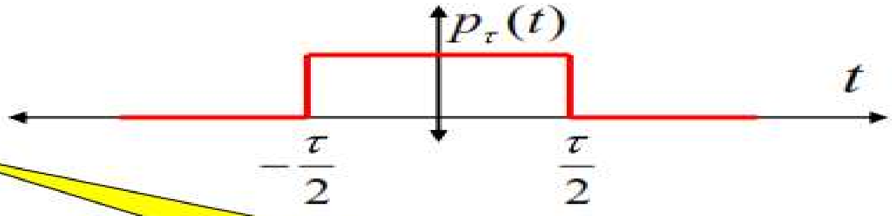
$$Z(\omega) = 6\pi\delta(\omega) + \frac{2}{jw}$$

Find the Fourier Transform of the following Function

Example: FT of a Rectangular pulse

τ = pulse width

Given: a rectangular pulse signal $p_\tau(t)$



Recall: we use this symbol to indicate a rectangular pulse with width τ

Find: $P_\tau(\omega)$… the FT of $p_\tau(t)$

Note the Notational Convention: lower-case for time signal and corresponding upper-case for its FT

Solution:

Note that

$$p_\tau(t) = \begin{cases} 1, & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0, & otherwise \end{cases}$$

Now apply the definition of the FT:

$$P_\tau(\omega) = \int_{-\infty}^{\infty} p_\tau(t) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

Limit integral to where $p_\tau(t)$ is non-zero… and use the fact that it is 1 over that region

$$= \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{\frac{-\tau}{2}}^{\frac{\tau}{2}} = \frac{2}{\omega}\left[\frac{e^{j\frac{\omega\tau}{2}} - e^{-j\frac{\omega\tau}{2}}}{j2}\right]$$

Artificially inserted 2 in numerator and denominator

$$= \sin\left(\frac{\omega\tau}{2}\right)$$

Use Euler's Formula

$$P_\tau(\omega) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

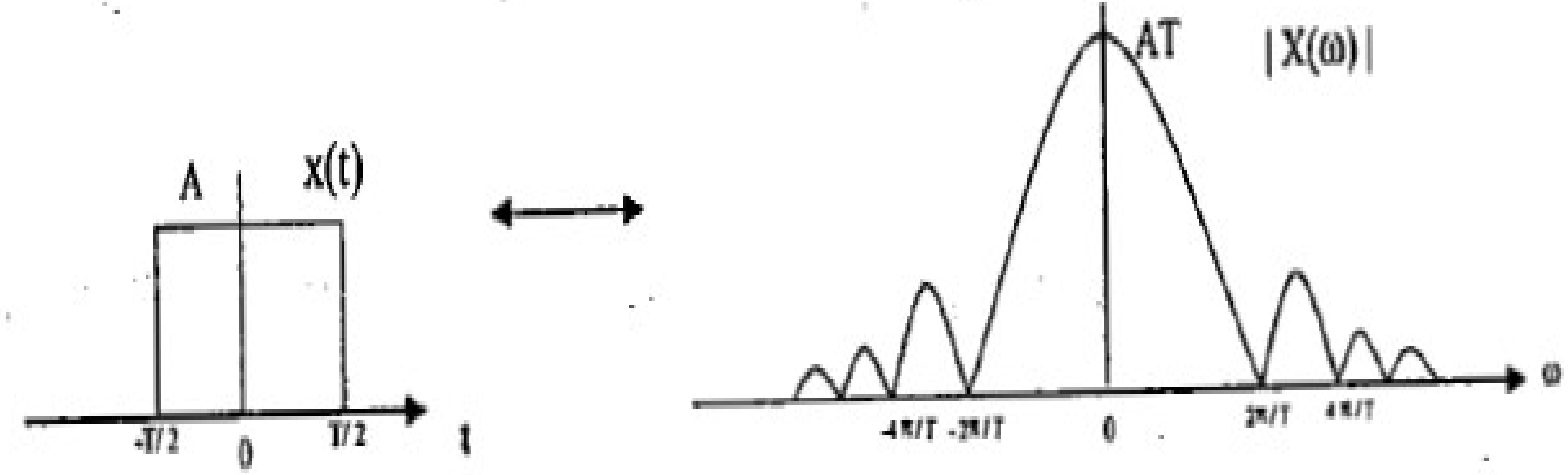
sin goes up and down between -1 and 1

1/ω decays down as |ω| gets big… this causes the overall function to decay down

Fourier Transform of Rectangular or Gate Function

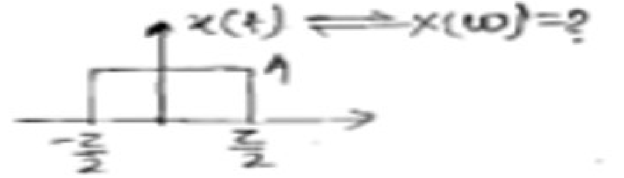
4) Rectangular (or) Gate function:-

$$x(t) = A\,\text{rect.}(t/T) \text{ (or) } A\,\pi(t/T) \longleftrightarrow A\,T\,\text{SinC}(\omega T/2)$$

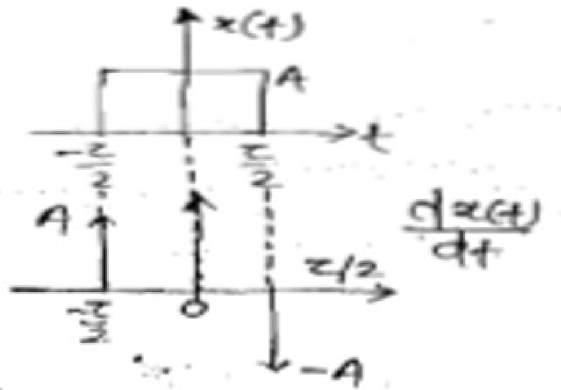


Find the Fourier Transform of the following Function

Q:



Sol:



$$\frac{dx(t)}{dt} = A\delta\left(t+\frac{\tau}{2}\right) - A\delta\left(t-\frac{\tau}{2}\right)$$

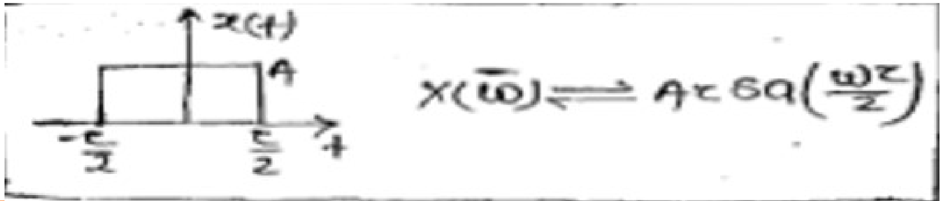
$$j\omega X(\omega) = Ae^{j\omega\frac{\tau}{2}} - Ae^{-j\omega\frac{\tau}{2}}$$

$$X(\omega) = \frac{A}{j\omega}\left[e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right]$$

$$= \frac{A}{j\omega} \text{ X } 2j \text{ X } \sin\left(\frac{\omega\tau}{2}\right)$$

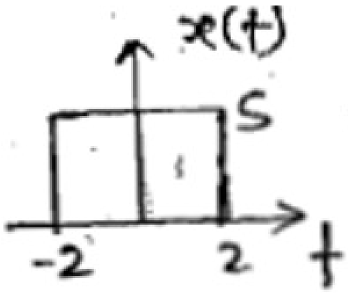
$$= \frac{2A}{\omega} \text{ X } \left[\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}\right] \text{ X } \frac{\omega\tau}{2}$$

$$= A\tau \text{ X } \mathrm{sa}\left(\frac{\omega\tau}{2}\right)$$



Find the Fourier Transform of the following Functions

Q:



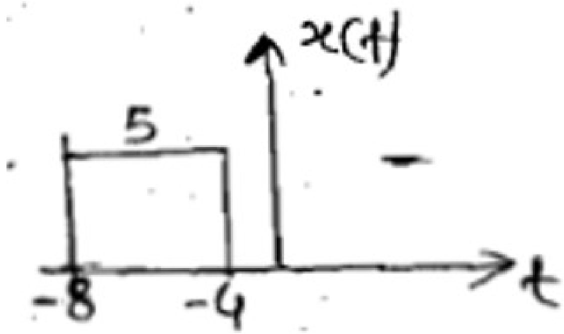
Sol:

$$X(\omega) = A\tau \text{ X } sa\left(\frac{\omega\tau}{2}\right)$$

$$= 5 \text{ X } 4 \text{ X } sa\left(\frac{\omega 4}{2}\right)$$

$$X(\omega) = 20 \text{ sa}(2\omega)$$

Q:



Sol:

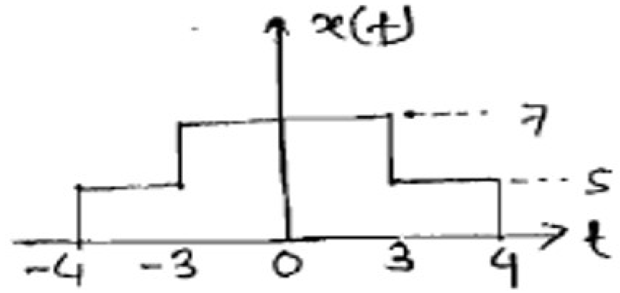
$$y(t) = x(t+6)$$

$$Y(\omega) = X(\omega)e^{j\omega 6}$$

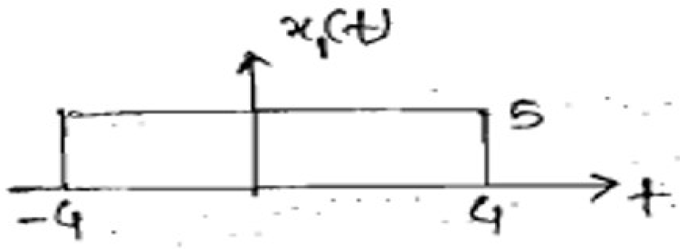
$$Y(\omega) = 20 \text{ sa}(2\omega)\, e^{j\omega 6}$$

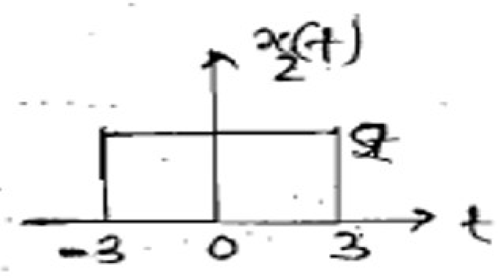
Find the Fourier Transform of the following Function

Q:



Sol:





$X_1(\omega) =$ 40 sa(4ω)

$X_2(\omega) =$ 12 sa(3ω)

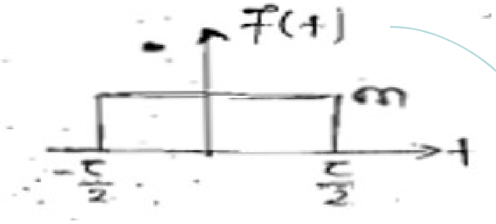
$$x(t) = x_1(t) + x_2(t)$$

$$X(\omega) = X_1(\omega) + X_2(\omega)$$

$$= 40\ \text{sa}(4\omega) + 12\ \text{sa}(3\omega)$$

Q: $x(t) = A_0 sa(t) \longrightarrow$ **Draw FT $X(\omega)$**

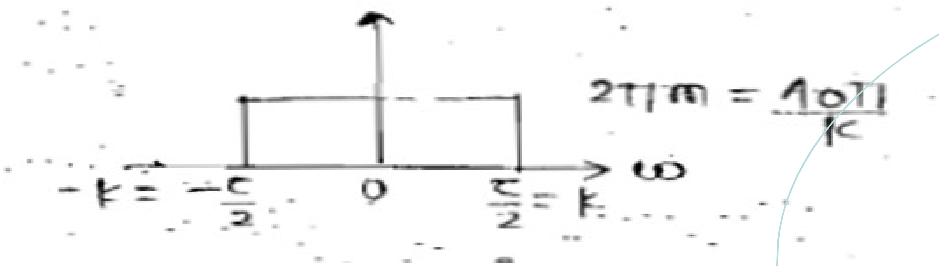
Sol:



$\longleftrightarrow$ $m\tau \, X sa\left(\frac{\omega\tau}{2}\right)$

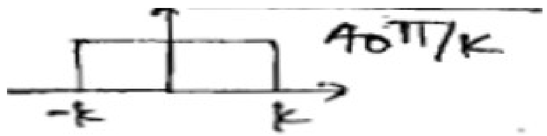
$(\omega = t)$ $(t = -\omega)$

$m\tau \, X sa\left(\frac{t\tau}{2}\right) \longleftrightarrow 2\pi f(-\omega)$



$$2\pi m = 2\pi \frac{A_0}{\tau} = 2\pi \frac{A_0}{2k} = \frac{\pi A_0}{k}$$

$A_0 \text{ sa}(kt) \Longleftrightarrow$



$m\tau \;\; sa\left(\frac{t\tau}{2}\right) \longleftrightarrow A_0 \text{ sa}(kt)$

$m\tau = A_0$ $\quad k = \frac{\tau}{2}$

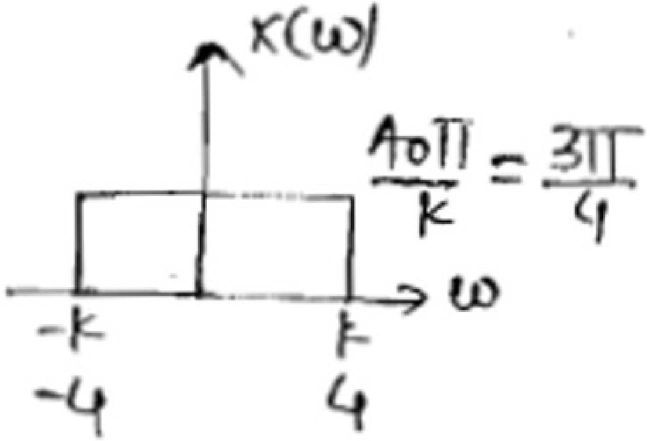
Find the Fourier Transform of the following Function

Q: $x(t) = 3\ \text{sa}(4t) \longleftrightarrow X(\omega)$

Sol:

Compare with $A_0\ \text{sa}(kt)$

$A_0=3 \quad k=4$

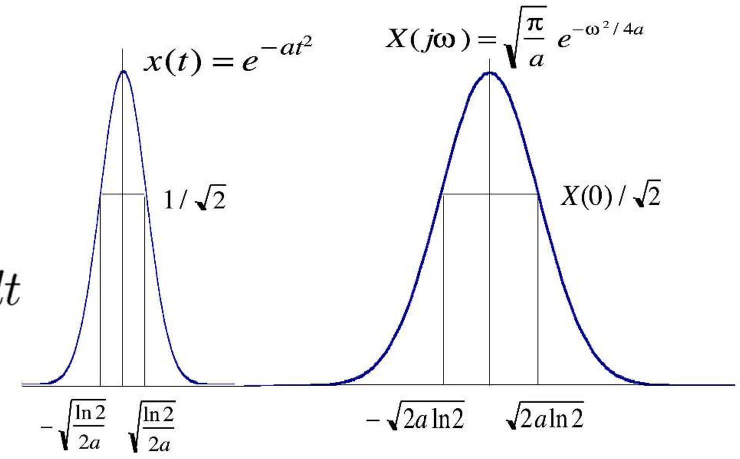


$$\frac{\pi A_0}{k} = \frac{3\pi}{4}$$

Fourier Transform of a Gaussian

$x(t) = e^{-at^2}$ — A Gaussian, important in probability,optics, etc.

$$
\begin{aligned}
X(j\omega) & \\
&= \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} e^{-a\left[t^2 + j\frac{\omega}{a}t + \left(\frac{j\omega}{2a}\right)^2\right] + a\left(\frac{j\omega}{2a}\right)^2} dt \\
&= \underbrace{\left[\int_{-\infty}^{\infty} e^{-a\left(t+\frac{j\omega}{2a}\right)^2} dt\right]}_{\sqrt{\pi}/a} \cdot e^{-\frac{\omega^2}{4a}} \\
&= \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}
\end{aligned}
$$



(Pulse width in t)•(Pulse width in ω)

$\Rightarrow \Delta t \cdot \Delta\omega \sim (1/a^{1/2}) \cdot (a^{1/2}) = 1$

Summary of Fourier Transform Properties

Signal $x(t)$	Fourier transform of a signal $x(t)$
$x(t)$	$X(w)$
$\delta(t)$	1
$u(t)$	$\frac{1}{jw}+\pi\delta(w)$
$sgn(t)$	$\frac{2}{jw}$
A_0	$2\pi A_0\delta(w)$
$e^{-at}u(t)$, $a>0$	$\frac{1}{a+jw}$
$e^{-a\|t\|}u(t)$, $a>0$	$\frac{2a}{a^2+w^2}$

Summary of Fourier Transform Properties

Signal $x(t)$	Fourier transform of a signal $x(t)$
$\cos w_0 t$	$\pi[\delta(w+w_0)+\delta(w-w_0)]$
$\sin w_0 t$	$\pi j[\delta(w+w_0)-\delta(w-w_0)]$
periodic signal	$2\pi \sum_{n=-\infty}^{n=\infty} c_n \delta(w-nw_0)$
$\sum \delta(t-nT_0)$	$w_0 \sum_{n=-\infty}^{\infty} \delta(w-nw_0)$
$e^{jw_0 t}$	$2\pi\delta(w-w_0)$
$e^{-jw_0 t}$	$2\pi\delta(w+w_0)$

Q: **The Fourier transform of $x(t) = u_1(t) + 2\delta(3-2t)$ is**

(where $u_1(t)$ the differentiation of an impulse)

a) $1 + e^{-j\frac{\omega}{2}}$ b) $2 + 3e^{-j\omega}$ c) $j\omega + e^{-j\frac{3\omega}{2}}$ d) $j\omega + e^{-j\frac{2\omega}{3}}$

Sol:

$$u_1(t) = \frac{d\delta(t)}{dt}$$

$$2\delta(3-2t) = 2\delta(2t-3) = 2 \times \frac{1}{2}\delta\left(t - \frac{3}{2}\right)$$

$$= \delta\left(t - \frac{3}{2}\right)$$

$$F\,T\,(u_1(t)) = j\omega$$

$$F\,T\,(x(t)) = j\omega + e^{-j\frac{3\omega}{2}}$$

Problem and Solution

The Fourier transform of $[\frac{\delta[t-t_0]}{a}]$ is

(a)$|a|e^{-j\omega t_0}$ (b) $1/|a|e^{j\omega t_0}$

(b) $\delta(\omega-\omega_0)e^{-j\omega t_0}$ (d) $e^{-j\omega t_0}$

Sol: $\delta(\frac{t-t_0}{a}) = |a|.e^{-j\omega t_0}$

Problem

Match the following

List I (function in time-domain)

A. Delta function
B. Gate function
C. Normalized Gaussian function
D. Sinusoidal function

List II (F.T. of the function)

1. Delta function
2. Gaussian function
3. Constant function
4. Sampling function

	A	B	C	D
(a)	1	2	4	3
(b)	3	4	2	1
(c)	1	2	2	3
(d)	3	4	4	1

Solution

Sol: Function In Time F.T. of Function

Delta function $\delta(t) \Rightarrow$ constant function

$\rightarrow$(3)

Gate function $\pi(t) \Rightarrow$ sampling function

$\rightarrow$(4)

Normalized Gaussian Function $\Rightarrow$ Gaussian function $\rightarrow$ (2)

Sinusoidal function $\Rightarrow$ Delta function$\rightarrow$(1)

Problem

Match the following

List I (signals)

(A) $g(t-2)$
(B) $t\, g(t)$
(C) $g(-t)$
(D) $G(3t + 1)$

List II (Transform)

(1) $j d/d\omega\, G(\omega)$
(2) $1/3\, G(\omega/3) e^{+j\omega/3}$
(3) $e^{-j2\omega} G(\omega)$
(4) $G(-\omega)$

Solution

4. Ans: (b)

Sol: Signals | F.T

Signals	F.T
g(t-2)	$G(\omega)e^{-j2\omega} \rightarrow (3)$
t g(t)	Frequency differentiation $j\frac{d}{d\omega}G(\omega) \rightarrow (1)$
g(-t)	Time reversal property $G(-\omega) \rightarrow (4)$
G(3t+1)	Scaling & shifting property $\frac{1}{3}G(\frac{\omega}{3})\, e^{\frac{+j}{3}} \rightarrow (2)$

Problem

Let $x(t) \leftrightarrow X(\omega) = \begin{cases} 1, & |\omega| < 1 \\ 0, & |\omega| > 1 \end{cases}$

Consider $y(t) = \frac{d^2x(t)}{dt^2}$. Then value of $\int_{-\infty}^{\infty} |y(t)|2 \; dt \; is$

(a) $\frac{3}{\pi}$ (b) $\frac{2}{3}$

(c) $\frac{1}{5\pi}$ (d) $\frac{1}{6\pi^2}$

Solution

Sol: $Y(\omega) = (j\omega)^2.X(\omega)$

$$\int_{-\infty}^{\infty} |y(t)|2 \ dt \ = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(\omega|2 \ dt$$

$$= \frac{1}{2\pi}\int_{-1}^{1} \omega^4 \ d\omega \ = \frac{1}{2\pi}\frac{\omega^5}{5}\Big|_{-1}^{1}$$

$$= \frac{1}{10}(2)$$

$$= \frac{1}{5\pi}$$

Problem

A Signal $x(t) = 8 - 8\cos^2(6\pi t)$ is passed through an ideal LPF. The filter blocks frequencies above 5Hz. Find the output?

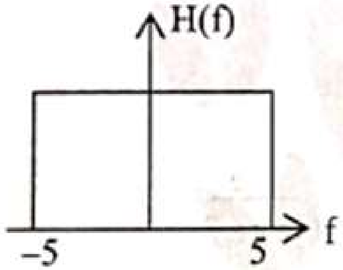
Solution

$x(t) = 8\text{-}8\cos^2(6\pi t)$

$= 8 - 8[\frac{1+\cos(12\pi t)}{2}]$

$x(t) = 8 - 4 - 4\cos(12\pi t)$

$x(t) = 4 - 4\cos(12\pi t)$



The frequencies of x(t) are 0,6Hz

6Hz frequency is not allowed only '0 Hz' is allowed y(t) = 4

Problem

The transfer function of a system is given by

$$H(\omega) = \frac{2+2j\omega}{4+4j\omega-\omega^2}$$

Find the output if input is $x(t) = e^{-t}\,u(t)$

Solution

Sol: $H(\omega) = \frac{2+2j\omega}{4+4j\omega+(j\omega)^2} = \frac{2+2j\omega}{(j\omega+2)^2}$

$x(t) = e^{-t}u(t)$

$X(\omega) = \frac{1}{1+j\omega}$

$Y(\omega) = H(\omega).X(\omega) = \frac{2+2j\omega}{(2+j\omega)^2}.\frac{1}{1+j\omega}$

$Y(\omega) = \frac{2}{(2+j\omega)^2}$ $\therefore y(t) = 2te^{-2t}u(t)$

Problem

Find the frequency and impulse response of a filter whose input – output relation is described by the following equation

$$Y(t) = x(t) - 2\int_{-\infty}^{t} y(\lambda)e^{(t-\lambda)}u(t-\lambda)\, d\lambda$$

Solution

Sol: y(t) = x(t) - $2\int_{-\infty}^{t} y(\lambda).\, e^{-(t-\tau)} u(t-\lambda) d\lambda$

y(t) = x(t) – 2[y(t)*e^{-t} u(t)]

Y(ω) = X(ω) – $2[\frac{Y(\omega)}{1+j\omega}]$

Y(ω)[1+ $\frac{2}{1+j\omega}$] = X(ω)

H(ω) = $\frac{Y(\omega)}{X(\omega)} = \frac{1+j\omega}{3+j\omega}$

H(ω) = 1 - $\frac{2}{3+j\omega}$

h(t) = δ(t) - 2e^{-3t}u(t)

Problem

The output and input of a causal LTI system are related by the Differential equation

$$\frac{d^2\, y(t)}{dt^2} + \frac{6dy(t)}{dt} + 8y(t) = 2x(t)$$

(a) Find the Impulse Response

(b) Find the response if $x(t) = te^{-2t}u(t)$

Solution

Sol: a) $\frac{d^2y(t)}{dt^2} + \frac{6\,dy(t)}{dt} + 8y(t) = 2x(t)$

$(j\omega)^2 \;\; Y(\omega) + \; 6j\omega Y(\omega + 8\,Y(\omega) = 2\,X(\omega)$

$H(\omega) \;=\; \frac{2}{(j\omega)2\;+6jw+} = \frac{2}{(j\omega+2)(j\omega+4)}$

$H(\omega) = \frac{A}{(j\omega+2)} + \frac{B}{(j\omega+4} = \frac{1}{(j\omega+2)} - \frac{1}{j\omega+4}$

$h(t) = (e^{-2t} \text{-}\; e^{-4t})u(t)$

b) $x(t) = te^{-2t}u(t)$ $\qquad$ $X(\omega) = \frac{1}{(2+j\omega)2}$

$Y(\omega) = H(\omega).X(\omega) = \frac{2}{(j\omega+2)3(j\omega+4)}$

$Y(\omega) = \frac{1/4}{(j\omega+2)} \;-\; \frac{1/2}{(j\omega+2)2} \;+\; \frac{1/2}{(j\omega+2)3} \;-\; \frac{1/4}{4+j\omega}$

$y(t) = (\;\frac{1}{4}e^{-2t} \text{-}\; \frac{t}{2}e^{-2t} + t^2\,e^{-2t} \;-\frac{1}{4}e^{-4t}\;)u(t)$

Problem

A LTI continuous–time system has frequency response $H(\omega)$, it is know that the input $x(t) = 1+4\cos(2\pi t)+8\sin(3\pi t - 90^0)$ produces the response

$y(t) = 2 – 2\sin(2\pi t)$. Then $H(\omega)$ at $\omega = 3\pi$ is

(a)0

(b) 1

(c) $(1/2)e^{-j\pi/2}$

(d) None of these

Solution

Sol: If input $x(t) = \cos\omega_0 t$

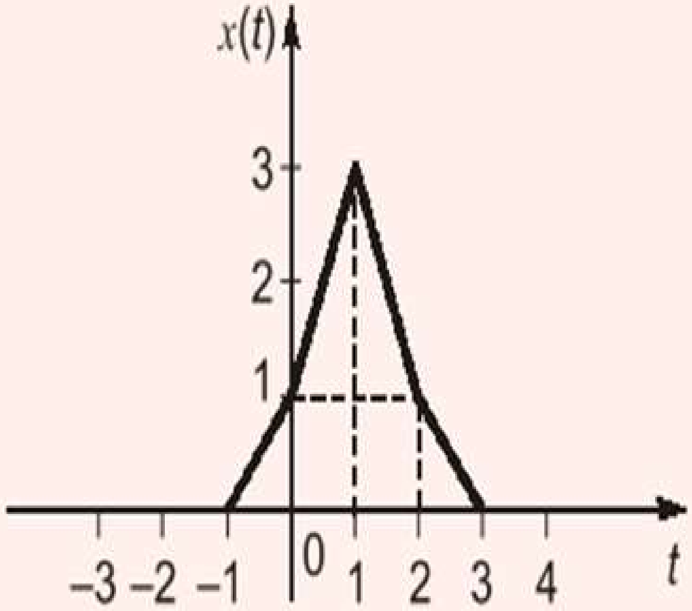
Frequency response $H(\omega)$, then output

$$y(t) = |H(\omega_0)|.\cos(\omega_0 t + \angle H(\omega_0))$$

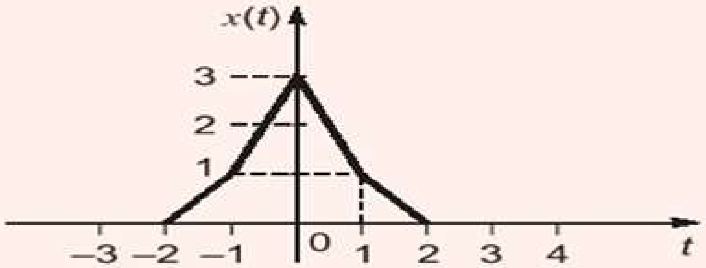
So $H(\omega)|_{\omega = 3\pi} = 0$. Because, there is no term of '3π' in y(t)

Problem

$X(\omega)$ is the Fourier transform of $x(t)$ shown below. The value of $\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$ (rounded off to two decimal places) is _____.



Solution



$$\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt = 2\pi \int_{-\infty}^{\infty} |y(t)|^2 dt$$

$$= 2 \times 2\pi \int_{-2}^{0} |y(t)|^2 dt$$

$$= 2 \times 2\pi \left[\int_{-2}^{-1} (t+2)^2 dt + \int_{-1}^{0} (2t+3)^2 dt \right]$$

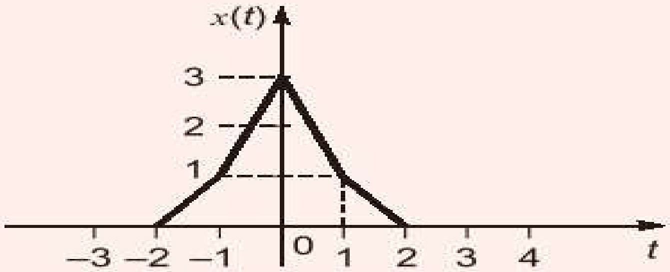
$$= 4\pi \left[\left\{ \frac{(t+2)^3}{3} \right\}_{-2}^{-1} + \left\{ \frac{(2t+3)^3}{3 \times 2} \right\}_{-1}^{0} \right]$$

$$= 4\pi \left[\frac{1-0}{3} + \frac{3^3-1}{6} \right] = 4\pi \left[\frac{1}{3} + \frac{26}{6} \right]$$

$$= 4\pi \times \left[\frac{1}{3} + \frac{26}{6} \right] = 4\pi \times \left[\frac{1}{3} + \frac{13}{3} \right]$$

$$= 4\pi \times \frac{14}{3} = \frac{56\pi}{3}$$

Ans. (58.61)



$$\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 \, dt = 2\pi \int_{-\infty}^{\infty} |y(t)|^2 \, dt$$

$$= 2 \times 2\pi \int_{-2}^{0} |y(t)|^2 \, dt$$

$$= 2 \times 2\pi \left[\int_{-2}^{-1} (t+2)^2 dt + \int_{-1}^{0} (2t+3)^2 dt \right]$$

$$= 4\pi \left[\left\{ \frac{(t+2)^3}{3} \right\}_{-2}^{-1} + \left\{ \frac{(2t+3)^3}{3 \times 2} \right\}_{-1}^{0} \right]$$

$$= 4\pi \left[\frac{1-0}{3} + \frac{3^3 - 1}{6} \right] = 4\pi \left[\frac{1}{3} + \frac{26}{6} \right]$$

$$= 4\pi \times \left[\frac{1}{3} + \frac{26}{6} \right] = 4\pi \times \left[\frac{1}{3} + \frac{13}{3} \right]$$

$$= 4\pi \times \frac{14}{3} = \frac{56\pi}{3}$$

Problem and Solution

The Fourier transform of a signal $h(t)$ is $H(j\omega) = (2\cos\omega)(\sin 2\omega)/\omega$. The value of $h(0)$ is

(A) 1/4

(B) 1/2

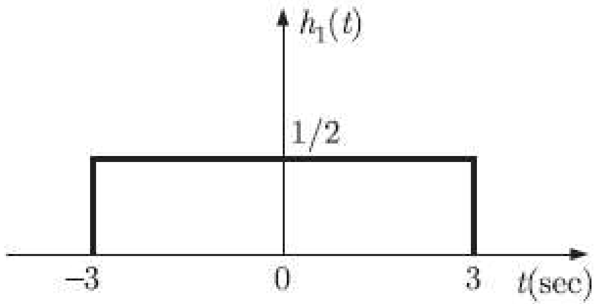
(C) 1

(D) 2

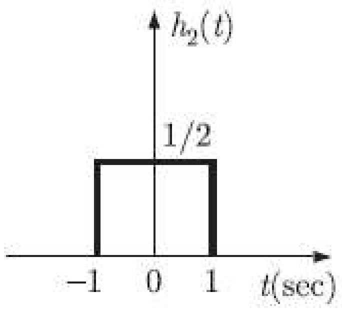
Option (C) is correct.

$$H(j\omega) = \frac{(2\cos\omega)(\sin 2\omega)}{\omega} = \frac{\sin 3\omega}{\omega} + \frac{\sin\omega}{\omega}$$

We know that inverse Fourier transform of $\sin c$ function is a rectangular function.



$$\xleftrightarrow{\mathcal{F}} \quad \frac{\sin 3\omega}{\omega}$$



$$\xleftrightarrow{\mathcal{F}} \quad \frac{\sin \omega}{\omega}$$

So, inverse Fourier transform of $H(j\omega)$

$$h(t) = h_1(t) + h_2(t)$$

$$h(0) = h_1(0) + h_2(0) = \frac{1}{2} + \frac{1}{2} = 1$$

Problem and Solution

The signal $x(t)$ is described by

$$x(t) = \begin{cases} 1 & \text{for} -1 \leq t \leq +1 \\ 0 & \text{otherwise} \end{cases}$$

Two of the angular frequencies at which its Fourier transform becomes zero are

(A) $\pi,\ 2\pi$

(B) $0.5\pi,\ 1.5\pi$

(C) $0,\ \pi$

(D) $2\pi,\ 2.5\pi$

Problem and Solution

Option (A) is correct.

We have $x(t) = \begin{cases} 1 & \text{for} -1 \leq t \leq +1 \\ 0 & \text{otherwise} \end{cases}$

Fourier transform is

$$\int_{-\infty}^{\infty} e^{-j\omega t} x(t)\, dt = \int_{-1}^{1} e^{-j\omega t} 1 dt = \frac{1}{-j\omega}[e^{-j\omega t}]_{-1}^{1}$$

$$= \frac{1}{-j\omega}(e^{-j\omega} - e^{j\omega}) = \frac{1}{-j\omega}(-2j\sin\omega) = \frac{2\sin\omega}{\omega}$$

This is zero at $\omega = \pi$ and $\omega = 2\pi$

Statement for Linked Answer Question

The impulse response $h(t)$ of linear time - invariant continuous time system is given by $h(t) = \exp(-2t)\,u(t)$, where $u(t)$ denotes the unit step function.

Q. The frequency response $H(\omega)$ of this system in terms of angular frequency ω, is given by $H(\omega)$

(A) $\frac{1}{1+j2\omega}$

(B) $\frac{\sin\omega}{\omega}$

(C) $\frac{1}{2+j\omega}$

(D) $\frac{j\omega}{2+j\omega}$

Q. The output of this system, to the sinusoidal input $x(t) = 2\cos 2t$ for all time t, is

(A) 0

(B) $2^{-0.25}\cos(2t - 0.125\pi)$

(C) $2^{-0.5}\cos(2t - 0.125\pi)$

(D) $2^{-0.5}\cos(2t - 0.25\pi)$

Problem and Solution

Sol. Option (C) is correct.

$$h(t) = e^{-2t}u(t)$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-2t} e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(2+j\omega)t}\, dt = \frac{1}{(2+j\omega)}$$

Sol. Option (D) is correct.

$$H(j\omega) = \frac{1}{(2+j\omega)}$$

The phase response at $\omega = 2$ rad/sec is

$$\angle H(j\omega) = -\tan^{-1}\frac{\omega}{2} = -\tan^{-1}\frac{2}{2} = -\frac{\pi}{4} = -0.25\pi$$

Magnitude response at $\omega = 2$ rad/sec is

$$|H(j\omega)| = \sqrt{\frac{1}{2^2 + w^2}} = \frac{1}{2\sqrt{2}}$$

Input is $\quad x(t) = 2\cos(2t)$

Output is

$$= \frac{1}{2\sqrt{2}} \times 2\cos(2t - 0.25\pi)$$

$$= \frac{1}{\sqrt{2}}\cos[2t - 0.25\pi]$$

Problem and Solution

Let $x(t) \longleftrightarrow X(j\omega)$ be Fourier Transform pair. The Fourier Transform of the signal $x(5t-3)$ in terms of $X(j\omega)$ is given as

(A) $\frac{1}{5}e^{-\frac{\beta\omega}{5}}X\left(\frac{j\omega}{5}\right)$

(B) $\frac{1}{5}e^{\frac{\beta\omega}{5}}X\left(\frac{j\omega}{5}\right)$

(C) $\frac{1}{5}e^{-\beta\omega}X\left(\frac{j\omega}{5}\right)$

(D) $\frac{1}{5}e^{\beta\omega}X\left(\frac{j\omega}{5}\right)$

Problem and Solution

Option (A) is correct.

$$x(t) \xleftrightarrow{F} X(j\omega)$$

Using scaling we have

$$x(5t) \xleftrightarrow{F} \frac{1}{5} X\left(\frac{j\omega}{5}\right)$$

Using shifting property we get

$$x\left[5\left(t-\frac{3}{5}\right)\right] \xleftrightarrow{F} \frac{1}{5} X\left(\frac{j\omega}{5}\right) e^{-\frac{j3\omega}{5}}$$

Problem and Solution

Let $x(n) = (\frac{1}{2})^n u(n), y(n) = x^2(n)$ and $Y(e^{j\omega})$ be the Fourier transform of $y(n)$ then $Y(e^{j0})$

(A) $\frac{1}{4}$

(B) 2

(C) 4

(D) $\frac{4}{3}$

Problem and Solution

Option (C) is correct.

$$F(s) = \frac{\omega_0}{s^2 + \omega^2}$$

$$L^{-1}F(s) = \sin \omega_o t$$

$$f(t) = \sin \omega_o t$$

Thus the final value is $-1 \leq f(\infty) \leq 1$

Problem and Solution

The output $y(t)$ of a linear time invariant system is related to its input $x(t)$ by the following equations

$$y(t) = 0.5x(t - t_d + T) + x(t - t_d) + 0.5x(t - t_d + T)$$

The filter transfer function $H(\omega)$ of such a system is given by

(A) $(1 + \cos \omega T)\, e^{-j\omega t_d}$

(B) $(1 + 0.5 \cos \omega T)\, e^{-j\omega t_d}$

(C) $(1 - \cos \omega T)\, e^{-j\omega t_d}$

(D) $(1 - 0.5 \cos \omega T)\, e^{-j\omega t_d}$

Problem and Solution

Option (A) is correct.

$$y(t) = 0.5x(t - t_d + T) + x(t - t_d) + 0.5x(t - t_d - T)$$

Taking Fourier transform we have

$$Y(\omega) = 0.5e^{-j\omega(-t_d+T)}X(\omega) + e^{-j\omega t_d}X(\omega) + 0.5e^{-j\omega(-t_d-T)}X(\omega)$$

or
$$\frac{Y(\omega)}{X(\omega)} = e^{-j\omega t_d}[0.5e^{j\omega T} + 1 + 0.5e^{-j\omega T}]$$

$$= e^{-j\omega t_d}[0.5\,(e^{j\omega T} + e^{-j\omega T}) + 1] = e^{-j\omega t_d}[\cos\omega T + 1]$$

or
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = e^{-j\omega t_d}(\cos\omega T + 1)$$

Problem and Solution

For a signal $x(t)$ the Fourier transform is $X(f)$. Then the inverse Fourier transform of $X(3f+2)$ is given by

(A) $\frac{1}{2}x\left(\frac{t}{2}\right)e^{j3\pi t}$

(B) $\frac{1}{3}x\left(\frac{t}{3}\right)e^{-\frac{j4\pi t}{3}}$

(C) $3x(3t)\,e^{-j4\pi t}$

(D) $x(3t+2)$

Problem and Solution

Option (B) is correct.

$$x(t) \xleftrightarrow{F} X(f)$$

Using scaling we have

$$x(at) \xleftrightarrow{F} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Thus $$x\left(\frac{1}{3}f\right) \xleftrightarrow{F} 3X(3f)$$

Using shifting property we get

$$e^{-j2\pi f_0 t} x(t) = X(f + f_0)$$

Thus $$\frac{1}{3} e^{-j\frac{4}{3}\pi t} x\left(\frac{1}{3}t\right) \xleftrightarrow{F} X(3f + 2)$$

$$e^{-j2\pi\frac{2}{3}t} x\left(\frac{1}{3}t\right) \xleftrightarrow{F} 3X(3(f + \tfrac{2}{3}))$$

$$\frac{1}{3} e^{-j\pi\frac{4}{3}t} x\left(\frac{1}{3}t\right) \xleftrightarrow{F} X[3(f + \tfrac{2}{3})]$$

Problem and Solution

The Fourier transform of a conjugate symmetric function is always

(A) imaginary

(B) conjugate anti-symmetric

(C) real

(D) conjugate symmetric

Option (C) is correct.

The Fourier transform of a conjugate symmetrical function is always real.

Problem and Solution

Let $x(t)$ be the input to a linear, time-invariant system. The required output is $4\pi(t-2)$. The transfer function of the system should be

(A) $4e^{j4\pi f}$ (B) $2e^{-j8\pi f}$

(C) $4e^{-j4\pi f}$ (D) $2e^{j8\pi f}$

Problem and Solution

Option (C) is correct.

$$y(t) = 4x(t-2)$$

Taking Fourier transform we get

$$Y(e^{j2\pi f}) = 4e^{-j2\pi f 2} X(e^{j2\pi f})$$ Time Shifting property

or $$\frac{Y(e^{j2\pi f})}{X(e^{j2\pi f})} = 4e^{-4j\pi f}$$

Thus $$H(e^{j2\pi f}) = 4e^{-4j\pi f}$$

Problem and Solution

The Fourier transform $F\{e^{-t}u(t)\}$ is equal to $\frac{1}{1+j2\pi f}$. Therefore, $F\left\{\frac{1}{1+j2\pi t}\right\}$ is

(A) $e^{f}u(f)$

(B) $e^{-f}u(f)$

(C) $e^{f}u(-f)$

(D) $e^{-f}u(-f)$

Option (C) is correct.

From the duality property of fourier transform we have

If $\quad x(t) \xleftrightarrow{FT} X(f)$

Then $\quad X(t) \xleftrightarrow{FT} x(-f)$

Therefore if $\quad e^{-t}u(t) \xleftrightarrow{FT} \frac{1}{1+j2\pi f}$

Then $\quad \frac{1}{1+j2\pi t} \xleftrightarrow{FT} e^{f}u(-f)$

Option (C) is correct.

From the duality property of fourier transform we have

If $$x(t) \xleftrightarrow{FT} X(f)$$

Then $$X(t) \xleftrightarrow{FT} x(-f)$$

Therefore if $$e^{-t}u(t) \xleftrightarrow{FT} \frac{1}{1+j2\pi f}$$

Then $$\frac{1}{1+j2\pi t} \xleftrightarrow{FT} e^{f}u(-f)$$

Problem and Solution

The Fourier Transform of the signal $x(t) = e^{-3t^2}$ is of the following form, where A and B are constants :

(A) $Ae^{-B|f|}$

(B) Ae^{-Bf^2}

(C) $A + B|f|^2$

(D) Ae^{-Bf}

Option (B) is correct.

Since normalized Gaussion function have Gaussion FT

Thus $$e^{-at^2} \overset{FT}{\longleftrightarrow} \sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi^2 f^2}{a}}$$

Problem and Solution

If $[f(t)] = F(s)$, then $[f(t-T)]$ is equal to

(A) $e^{sT}F(s)$

(B) $e^{-sT}F(s)$

(C) $\dfrac{F(s)}{1-e^{sT}}$

(D) $\dfrac{F(s)}{1-e^{-sT}}$

Option (B) is correct.

If $$\mathcal{L}[f(t)] = F(s)$$

Applying time shifting property we can write

$$\mathcal{L}[f(t-T)] = e^{-sT}F(s)$$

Problem and Solution

A signal $x(t)$ has a Fourier transform $X(\omega)$. If $x(t)$ is a real and odd function of t, then $X(\omega)$ is

(A) a real and even function of ω

(B) a imaginary and odd function of ω

(C) an imaginary and even function of ω

(D) a real and odd function of ω

Option (A) is correct.

Problem and Solution

The Fourier transform of a real valued time signal has

(A) odd symmetry
(B) even symmetry
(C) conjugate symmetry
(D) no symmetry

Option (C) is correct.

The conjugation property allows us to show if $x(t)$ is real, then $X(j\omega)$ has conjugate symmetry, that is

$$X(-j\omega) = X^*(j\omega) \qquad [x(t) \text{ real}]$$

Proof :

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

replace ω by $-\omega$ then

$$X(-j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt$$

$$X^*(j\omega) = \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right]^* = \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt$$

if $x(t)$ real $x^*(t) = x(t)$

then $$X^*(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt = X(-j\omega)$$

Sources, References and Acknowledgement

i) Lecture slides of Michael D. Adams

ii) Lecture slides of Prof. Paul Cuff

iii) A.V. Oppenheim, A.S. Willsky and S. H. Nawab, “Signals and Systems”, PHI, 2nd Edition, 1997.

iv) GATE Previous Questions with Solutions, Nodia Publications

v) MIT Open Courseware http://ocw.mit.edu

vi) https://www.aceenggacademy.com/

vii) https://www.madeeasy.in/

- Disclaimer: The material presented in this presentation is taken from various standard Textbooks and Internet Resources and the presenter is acknowledging all the authors.