Lecture 6

Electromagnetic Field Theory

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"Our thoughts and feelings have electromagnetic reality. Manifest wisely."

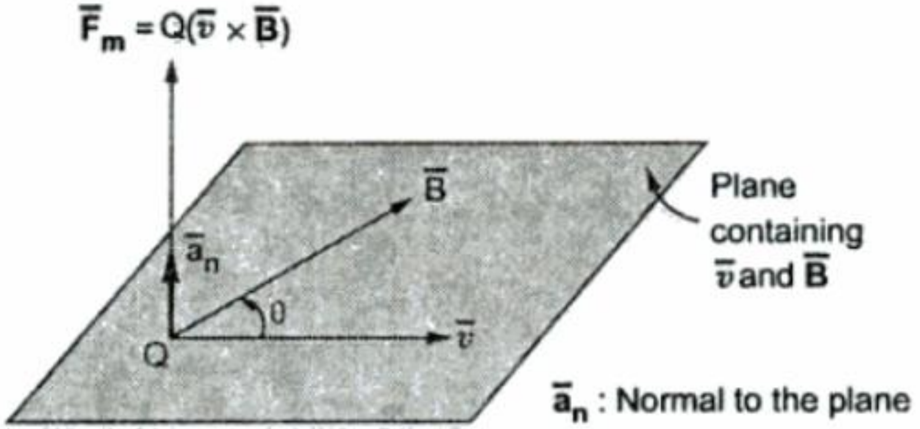
Topics

1. Magnetic Force
2. Lorentz force
3. Faraday’s law

Lorentz Force

$$\boxed{\bar{F}_e = Q\,\bar{E} \;\; N}$$

$$\boxed{\bar{F}_m = Q\,\bar{v} \times \bar{B} \;\; N}$$



The total force on a moving charge in the presence of both electric and magnetic fields

$$\vec{F} = \underbrace{q\vec{E}}_{\text{Electric force}} + \underbrace{q\vec{v} \times \vec{B}}_{\text{Magnetic force}}$$

Above equation is called **Lorentz Force Equation** which relates mechanical force to the electrical force. If the mass of the charge is m, then we can write,

$$\boxed{\bar{F} = m\,\bar{a} = m\frac{d\bar{v}}{dt} = Q\left(\bar{E} + \bar{v} \times \bar{B}\right) \;\; N}$$

Problem

A point charge of Q = – 1.2 C has velocity $\bar{v} = (5\bar{a}_x + 2\bar{a}_y - 3\bar{a}_z)$ *m/s. Find the magnitude of the force exerted on the charge if,*

a) $\bar{E} = -18\bar{a}_x + 5\bar{a}_y - 10\bar{a}_z$ *V/m,*

b) $\bar{B} = -4\bar{a}_x + 4\bar{a}_y + 3\bar{a}_z$ *T,*

c) Both are present simultaneous.

Solution : a) The electric force exerted by $\bar{E}$ on charge Q is given by,

$$\bar{F}_e = Q\bar{E}$$

$$= -1.2[-18\bar{a}_x + 5\bar{a}_y - 10\bar{a}_z]$$

$$= 21.6\,\bar{a}_x - 6\,\bar{a}_y + 12\,\bar{a}_z \text{ N}$$

Thus the magnitude of the electric force is given by

$$|\bar{F}_e| = \sqrt{(21.6)^2 + (-6)^2 + (12)^2} = 25.4275 \text{ N}$$

Problem

b) The magnetic force exerted by $\bar{B}$ on charge Q is given by,

$$\begin{aligned}
\bar{F}_m &= Q\,\bar{v}\times\bar{B} \\
&= -1.2\left[(5\bar{a}_x + 2\bar{a}_y - 3\bar{a}_z)\times(-4\bar{a}_x + 4\,\bar{a}_y + 3\bar{a}_z)\right] \\
&= (-6\,\bar{a}_x - 2.4\bar{a}_y + 3.6\bar{a}_z)\times(-4\bar{a}_z + 4\,\bar{a}_y + 3\,\bar{a}_z) \\
&= \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ -6 & -2.4 & 3.6 \\ -4 & 4 & 3 \end{vmatrix} \\
&= [-7.2 - 14.4]\,\bar{a}_x - [-18 + 14.4]\,\bar{a}_y + [-24 - 9.6]\,\bar{a}_z \\
&= (-21.6\,\bar{a}_x + 3.6\,\bar{a}_y - 33.6\,\bar{a}_z)\text{ N}
\end{aligned}$$

Thus, the magnitude of the magnetic force is given by

$$|\bar{F}_m| = \sqrt{(-21.6)^2 + (+3.6)^2 + (-33.6)^2} = \mathbf{40.1058\ N}$$

Problem

c) The total force exerted by both the fields ($\bar{E}$ and $\bar{B}$) on a charge is given by,

$$\bar{F} = \bar{F}_e + \bar{F}_m = Q\left(\bar{E} + \bar{v} \times \bar{B}\right) = Q\,\bar{E} + Q\,\bar{v} \times \bar{B}$$

$$= \left[\left(21.6\,\bar{a}_x - 6\,\bar{a}_y + 12\,\bar{a}_z\right) + \left(-21.6\,\bar{a}_x + 3.6\,\bar{a}_y - 33.6\,\bar{a}_z\right)\right]$$

$$= \left(0\,\bar{a}_x - 2.4\,\bar{a}_y - 21.6\,\bar{a}_z\right)\ \text{N}$$

Thus, the magnitude of the total force exerted is given by

$$\left|\bar{F}\right| = \sqrt{(0)^2 + (-2.4)^2 + (-21.6)^2} = \mathbf{21.7329\ \ N}$$

Force on a differential Current Element

The force exerted on a differential element of charge dQ moving in a steady magnetic field is given by,

$$d\bar{F} = dQ\,\bar{v} \times \bar{B} \quad N$$

The current density $\bar{J}$ can be expressed interms of velocity of a volume charge density as,

$$\bar{J} = \rho_v\,\bar{v}$$

But the differential element of charge can be expressed in terms of the volume charge density as,

$$dQ = \rho_v\,dv$$

$$\therefore \quad d\bar{F} = \rho_v\,dv\,\bar{v} \times \bar{B}$$

Force on a differential Current Element

Expressing $d\bar{F}$ interms of $\bar{J}$

$$d\bar{F} = \bar{J} \times \bar{B} \; dv$$

the relationship between current element as,

$$\bar{J}\, dv = \bar{K}\, dS = I\, d\bar{L}$$

Then the force exerted on a surface current density is given by,

$$d\bar{F} = \bar{K} \times \bar{B} \; dS$$

Similarly the force exerted on a differential current element is given by,

$$d\bar{F} = \left(I\, d\bar{L} \times \bar{B}\right)$$

Integrating over a volume, the force is given by,

$$\bar{F} = \int_{vol} \bar{J} \times \bar{B}\, dv$$

$$\bar{F} = \int_{S} \bar{K} \times \bar{B} \; dS \qquad \boxed{\bar{F} = \oint I\, d\bar{L} \times \bar{B}}$$

Problem

A conductor 6m long, lies along z-direction with a current of 2A in $\bar{a}_z$ direction. Find the force experienced by conductor if $\bar{B} = 0.08\ \bar{a}_x$ (T).

Solution : A force exerted on current carrying conductor in a magnetic field is given by

$$\bar{F} = I\ d\bar{L} \times \bar{B}$$

$$\therefore \quad \bar{F} = 2(6\bar{a}_z) \times (0.08\bar{a}_x)$$

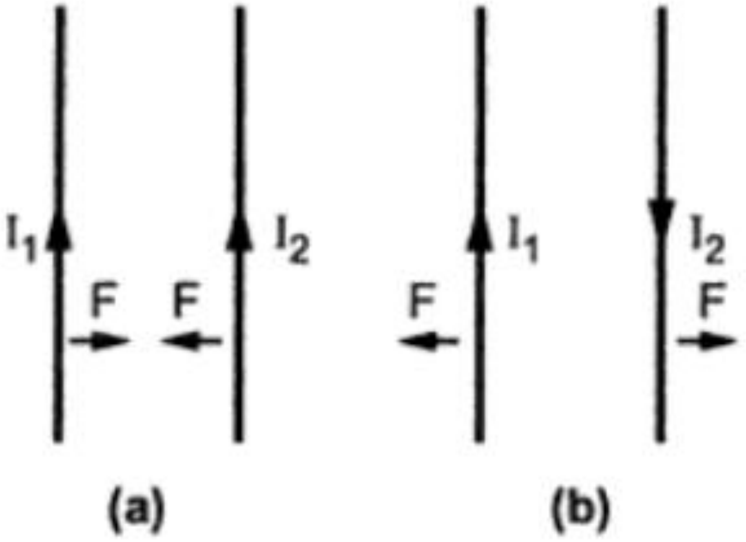
$$\therefore \quad \bar{F} = 12\ \bar{a}_z \times 0.08\ \bar{a}_x$$

$$\therefore \quad \bar{F} = 0.96\ \bar{a}_y\ \ N \qquad \ldots\ \bar{a}_z \times \bar{a}_x = \bar{a}_y$$

Force between differential Current Elements

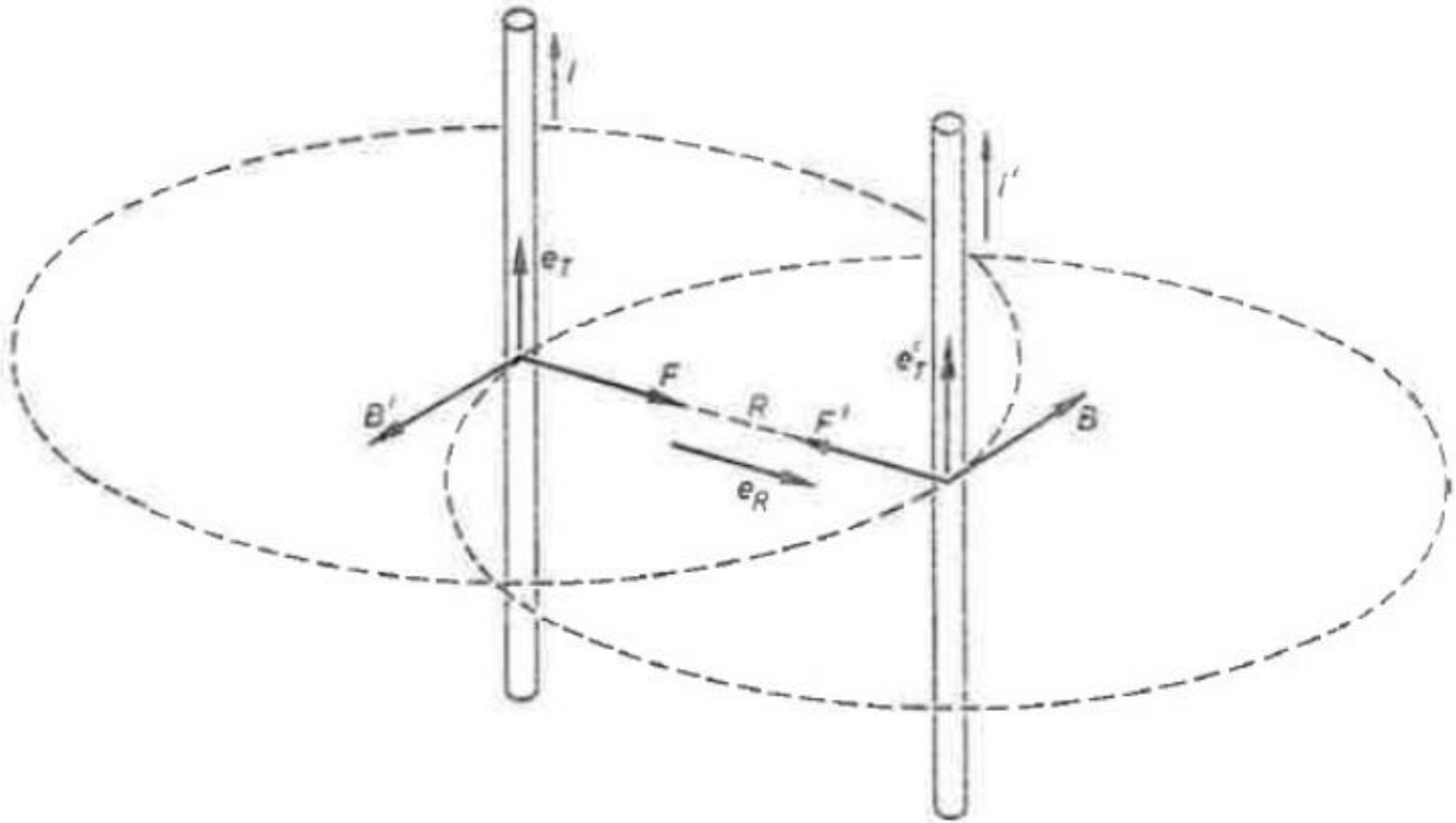
Now consider that two current carrying conductors are placed parallel to each other. Each of this conductor produces its own flux around it. So when such two conductors are placed closed to each other, there exists a force due to the interaction of two fluxes. The force between such parallel current carrying conductors depends on the directions of the two currents. If the directions of both the currents are same, then the conductors experience a force of attraction

And if the directions of two currents are opposite to each other, then the conductors experience a force of repulsion



Force between two parallel current carrying conductors

Force between differential Current Elements



Force between differential Current Elements

Let us now consider two current elements $I_1\, d\bar{L}_1$ and $I_2 d\bar{L}_2$ the directions of I_1 and I_2 are same.

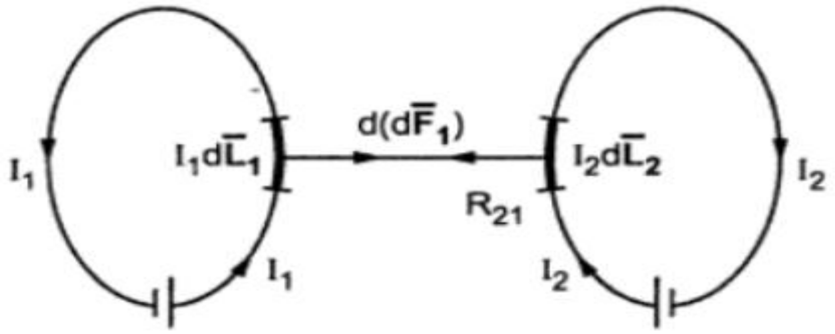
Both the current elements produce their own magnetic fields. As the currents are flowing in the same direction through the elements, the force $d(d\bar{F}_1)$ exerted on element $I_1 d\bar{L}_1$ due to the magnetic field $d\bar{B}_2$ produced by other element $I_2 d\bar{L}_2$ is the force of attraction.

From the equation of force the force exerted on a differential current element is given by,

$$d(d\bar{F}_1) = I_1 d\bar{L}_1 \times d\bar{B}_2$$

According to Biot-Savart's law, the magnetic field produced by current element $I_2 d\bar{L}_2$ is given by, for free space,

$$d\bar{B}_2 = \mu_0\, d\bar{H}_2 = \mu_0 \left[\frac{I_2 d\bar{L}_2 \times \bar{a}_{R21}}{4\pi R_{21}^2}\right]$$



Substituting value of $d\bar{B}_2$ in equation (1), we can write,

$$d(d\bar{F}_1) = \mu_0 \frac{I_1 d\bar{L}_1 \times (I_2 d\bar{L}_2 \times \bar{a}_{R21})}{4\pi R_{21}^2}$$

Force between differential Current Elements

$$\boxed{\bar{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\bar{L}_1 \times (d\bar{L}_2 \times \bar{a}_{R21})}{R_{21}^2}}$$

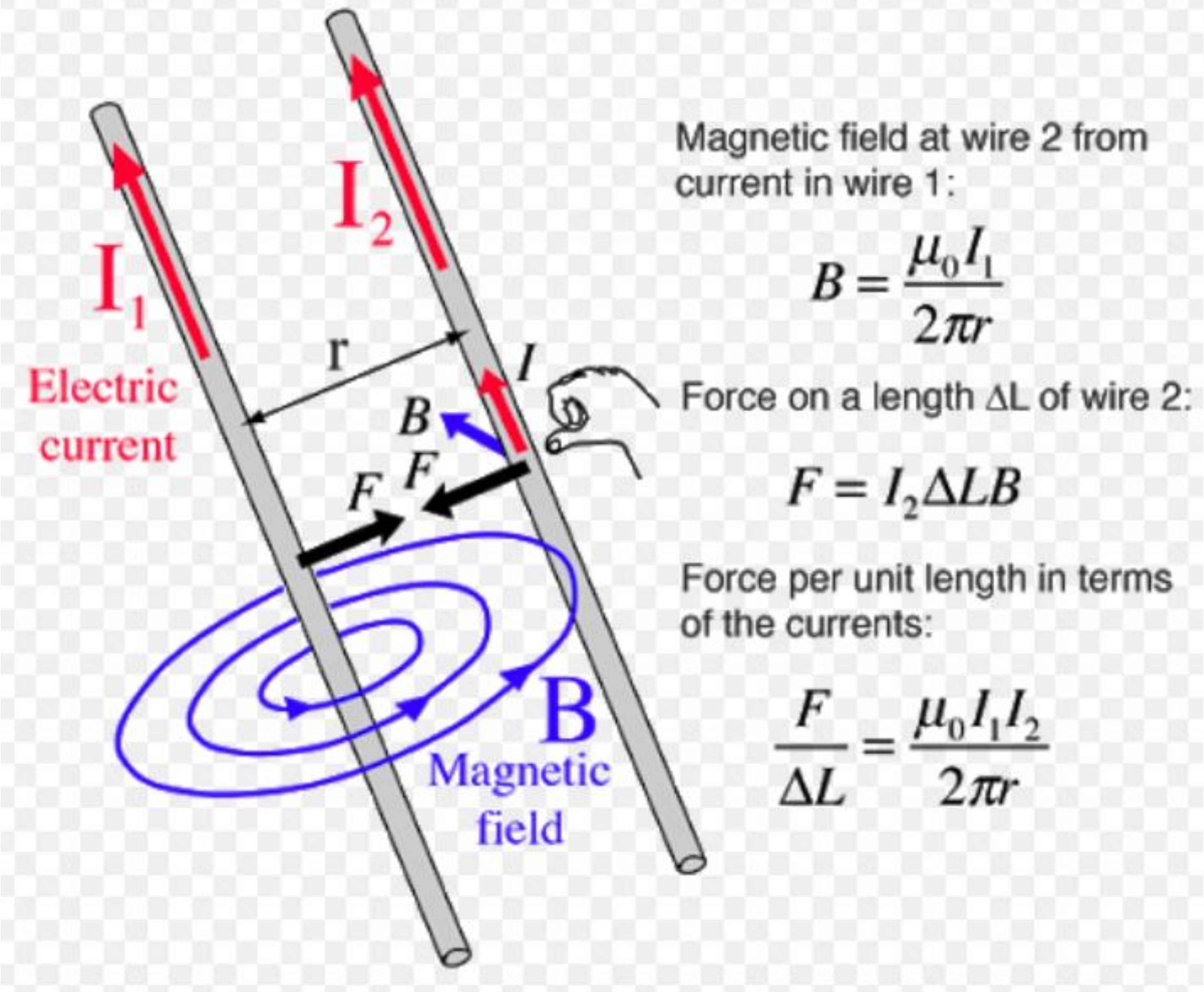
Exactly following same steps, we can calculate the force $\bar{F}_2$ exerted on the current element 2 due to the magnetic field $\bar{B}_1$ produced by the current element 1. Thus,

$$\boxed{\bar{F}_2 = \frac{\mu_0 I_2 I_1}{4\pi} \oint_{L_2} \oint_{L_1} \frac{d\bar{L}_2 \times (d\bar{L}_1 \times \bar{a}_{R12})}{R_{12}^2}}$$

$$\bar{F}_2 = -\bar{F}_1$$

Thus, above condition indicates that both the forces $\bar{F}_1$ and $\bar{F}_2$ obey Newton's third law that for every action there is equal and opposite reaction.

Force between differential Current Elements



Problem

A current element, $I_1 \Delta \bar{L}_1 = 10^{-5}\,\bar{a}_z$ A.m is located at $P_1(1,0,0)$ while a second element, $I_2 \Delta \bar{L}_2 = 10^{-5}(0.6\bar{a}_x - 2\,\bar{a}_y + 3\,\bar{a}_z)$ A.m is at $P_2(-1,0,0)$, both in free space. Find the vector force exerted on $I_2 \Delta \bar{L}_2$ by $I_1 \Delta \bar{L}_1$.

Solution : The magnetic field intensity at point P_1 due to $I_1 \Delta \bar{L}_1$ can be obtained using Biot-Savart's law as follows.

$$d\bar{H}_1 = \frac{I_1 \Delta \bar{L}_1 \times \bar{a}_{R12}}{4\pi (R_{12})^2}$$

The unit vector in the direction of $\bar{R}_{12}$ is drawn from P_1 to P_2.

$$\therefore \quad \bar{R}_{12} = (-1-1)\bar{a}_x = -2\,\bar{a}_x$$

$$R_{12} = |\bar{R}_{12}| = \sqrt{(-2)^2} = 2$$

$$\therefore \quad \bar{a}_{R12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{-2\,\bar{a}_x}{2} = -\,\bar{a}_x$$

Substituting values of $I_1 \Delta \bar{L}_1$, $\bar{a}_{R12}$ and R_{12} in the expression for $d\bar{H}_1$, we get,

$$d\bar{H}_1 = \frac{(10^{-5}\,\bar{a}_z)\times(-\bar{a}_x)}{4\pi(2)^2} = -\frac{10^{-5}}{16\pi}\,\bar{a}_y \quad \text{A/m}$$

Problem

Thus the magnetic flux density at point P_1 is given by,

$$d\bar{B}_1 = \mu_0\, d\bar{H}_1 = (4\pi\times10^{-7})\left(\frac{-10^{-5}}{16\pi}\,\bar{a}_y\right) = -0.25\times10^{-12}\,\bar{a}_y \text{ T}$$

Now the force exerted on $I_2\Delta\bar{L}_2$ due to $I_1\Delta\bar{L}_1$ is given by,

$$\begin{aligned}
F_2 &= I_2\Delta\bar{L}_2\times d\bar{B}_1\\
&= 10^{-5}(0.6\bar{a}_x - 2\,\bar{a}_y + 3\,\bar{a}_z)\times(-0.25\times10^{-12}\,\bar{a}_y)\\
&= 10^{-17}[(0.6\bar{a}_x - 2\,\bar{a}_y + 3\,\bar{a}_z)\times(-0.25\,\bar{a}_y)]\\
&= [-0.15\,\bar{a}_z + 0.75\,\bar{a}_x](10^{-17})\\
&= (7.5\,\bar{a}_x - 1.5\,\bar{a}_z)\,10^{-18}\text{ N}
\end{aligned}$$

Problem

Two wires carrying current in the same direction of 3 A and 6 A are placed with their axes 5 cm apart, free space permeability = $4\pi \times 10^{-7}$ H/m. Calculate the force between them in kg/m length.

Solution : Force between two parallel conductors is given by,

$$F = \frac{\mu I_1 \ I_2 \, l}{2\pi d}$$

d = Distance of separation = 5 cm = 5×10^{-2} m

$$I_1 = 3 \text{ A}$$

$$I_2 = 6 \text{ A}$$

$$l = \text{Length of conductors}$$

Hence force per unit meter length is given by,

$$\frac{F}{l} = \frac{\mu I_1 \ I_2}{2\pi d} = \frac{\mu_0 \ \mu_r \ I_1 \ I_2}{2\pi d}$$

For free space $\mu_r = 1$

$$\therefore \quad \frac{F}{l} = \frac{4 \times \pi \times 10^{-7} \times 3 \times 6}{2 \times \pi \times 5 \times 10^{-2}} = 7.2 \times 10^{-5} \text{ N/m}$$

$$\therefore \quad \frac{F}{l} = 72 \ \mu\text{N/m}$$

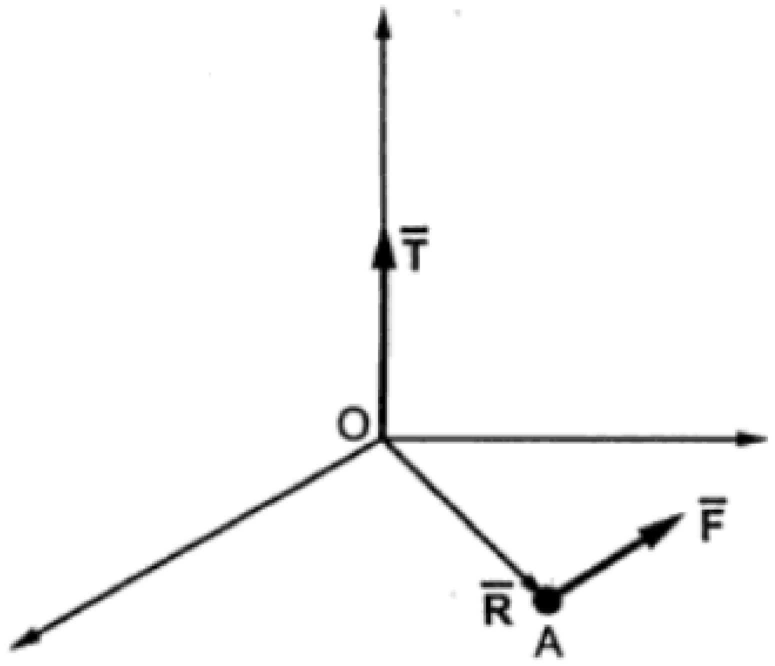
The force expressed in kg/m is given by,

$$\frac{F}{l} = \frac{72 \times 10^{-6} \text{ N/m}}{9.8} = 7.3469 \times 10^{-6} \text{ kg/m}$$

Magnetic Torque

The **moment of a force or torque** about a specified point is defined as the vector product of the moment arm $\bar{R}$ and the force $\bar{F}$. It is measured in newton meter (Nm).

$$\therefore \quad \boxed{\bar{T} = \bar{R} \times \bar{F} \text{ Nm}}$$



Magnetic Torque

Now consider that two forces namely $\bar{F}_1$ and $\bar{F}_2$ are applied at points A_1 and A_2 respectively. The arms for the two forces drawn from the origin be $\bar{R}_1$ and $\bar{R}_2$ respectively :

Assume that $\bar{F}_2 = -\bar{F}_1$. Then the total torque $\bar{T}$ about the origin due to the two forces is given by

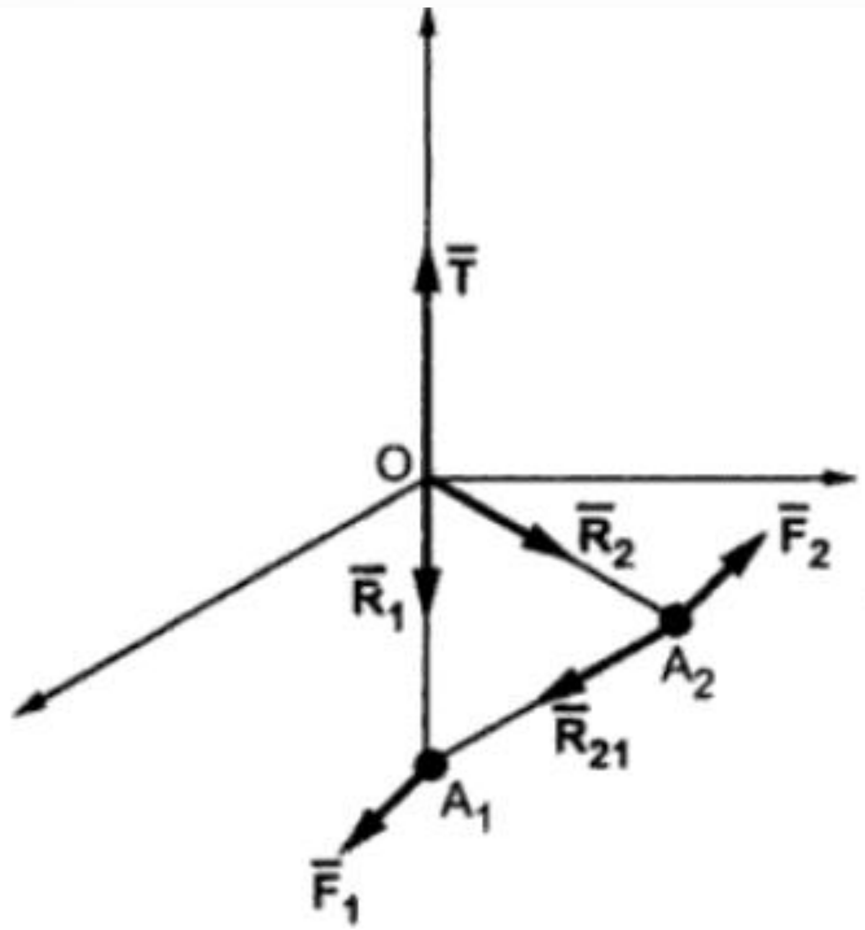
$$\bar{T} = \bar{R}_1 \times \bar{F}_1 + \bar{R}_2 \times \bar{F}_2$$

$$\therefore \quad \bar{T} = (\bar{R}_1 - \bar{R}_2) \times \bar{F}_1 \qquad \ldots \bar{F}_2 = -\bar{F}_1$$

$$\therefore \quad \bar{T} = \bar{R}_{21} \times \bar{F}_1$$

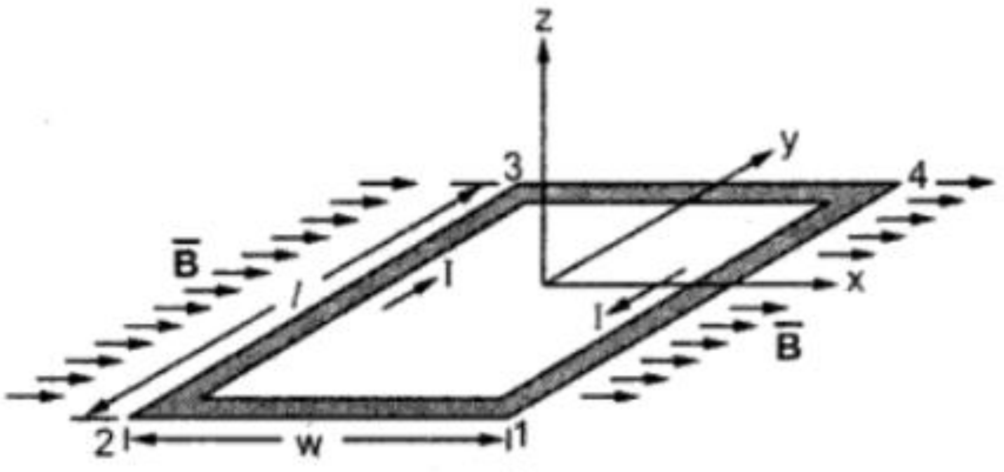
where $\bar{R}_{21} = \bar{R}_1 - \bar{R}_2$ is a vector joining A_2 to A_1

From above expression it is clear that when total force is zero, the torque is independent of the choice of the origin.



Magnetic Moment of Planar Coil

Let us consider a rectangular planer coil fo length l along y-axis and width w along x-axis. The coil is placed in the uniform magnetic field $\bar{B}$ which is positive x-direction



Rectangular planar coil in uniform magnetic field $\bar{B}$

Actually the current I is flowing in clockwise direction in closed path 1-2-3-4-1. As sides 3-4 and 1-2 are parallel to the direction of $\bar{B}$, no force will be exerted on those. The current flowing through the other sides i.e. side 2-3 and 1-4, is in either + y or − y-direction. So these sides contribute in force exerted on a planar coil as whole.

For side 2-3, the force exerted is given by,

$$\bar{F}_1 = I(l\bar{a}_y \times B\bar{a}_x) = -BIl\bar{a}_z$$

Similarly for side 4-1, the force exerted is given by,

$$\bar{F}_2 = I(-l\bar{a}_y \times B\bar{a}_x) = +BIl\bar{a}_z$$

For the current element along side 2-3 the moment arm is given by

$$\bar{R}_1 = -\frac{w}{2}\bar{a}_x$$

Similarly for the current element along side 4-1, the moment arm is given by

$$\bar{R}_2 = +\frac{w}{2}\bar{a}_x$$

Thus the total torque $\bar{T}$ about y-axis is given by

$$\bar{T} = \bar{T}_1 + \bar{T}_2 = \bar{R}_1 \times \bar{F}_1 + \bar{R}_2 \times \bar{F}_2$$

$$\therefore \quad \bar{T} = \left(-\frac{w}{2}\bar{a}_x\right)\times(-BIl\,\bar{a}_z)+\left(\frac{w}{2}\bar{a}_x\right)\times(BIl\,\bar{a}_z)$$

$$\therefore \quad \bar{T} = -\frac{w}{2}BIl\,\bar{a}_y - \frac{w}{2}BIl\,\bar{a}_y$$

$$\therefore \quad \bar{T} = BI(l\,w)\,(-\bar{a}_y)$$

But we can write Area (S) = length (l) × Width (W)

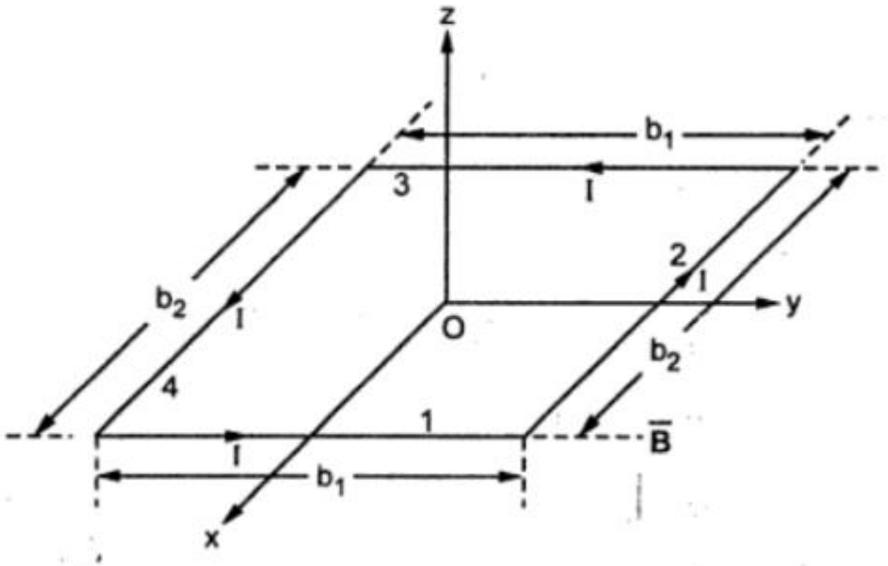
$$\therefore \quad \boxed{\bar{T} = BIS\,(-\bar{a}_y)}$$

This indicates that eventhough the total force exerted on the rectangular coil as a whole is zero, we get a torque along the axis of rotation. The expression is valid for all flat coils of arbitary shapes and also for any axis which is parallel to the axis of rotation i.e. y-axis.

Problem

A rectangular loop in the xy plane with sides b_1 and b_2 carrying a current I lies in a uniform magnetic field $B = \bar{a}_x B_x + \bar{a}_y B_y + \bar{a}_z B_z$. Determine the force and torque on the loop.

Solution : Consider a rectangular loop in the xy plane with sides b_1 and b_2



Sides 1 and 3 are of length b_1 and are parallel to y-axis with sides 2 and 4 are of length b_2 and are parallel to x-axis. The origin is at the centre of the loop.

The vector force on side 1 is given by,

$$\bar{F}_1 = I\, d\bar{L}_1 \times \bar{B}$$

Problem

$$= \left[I\left(b_1 \bar{a}_y\right) \times \left(B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z\right)\right]$$

$$= I b_1 \left[- B_x \bar{a}_z - B_z \bar{a}_x\right]$$

$$= I b_1 \left[B_z \bar{a}_x - B_x \bar{a}_z\right]$$

The vector force on the side 2 is given by,

$$\bar{F}_2 = I\, d\bar{L}_2 \times \bar{B}$$

$$= \left[I\left(-b_2 \bar{a}_x\right) \times \left(B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z\right)\right]$$

$$= - I\, b_2 \left[B_y \bar{a}_z - B_y \bar{a}_y\right]$$

$$= - I\, b_2 \left[- B_z \bar{a}_y + B_y \bar{a}_z\right]$$

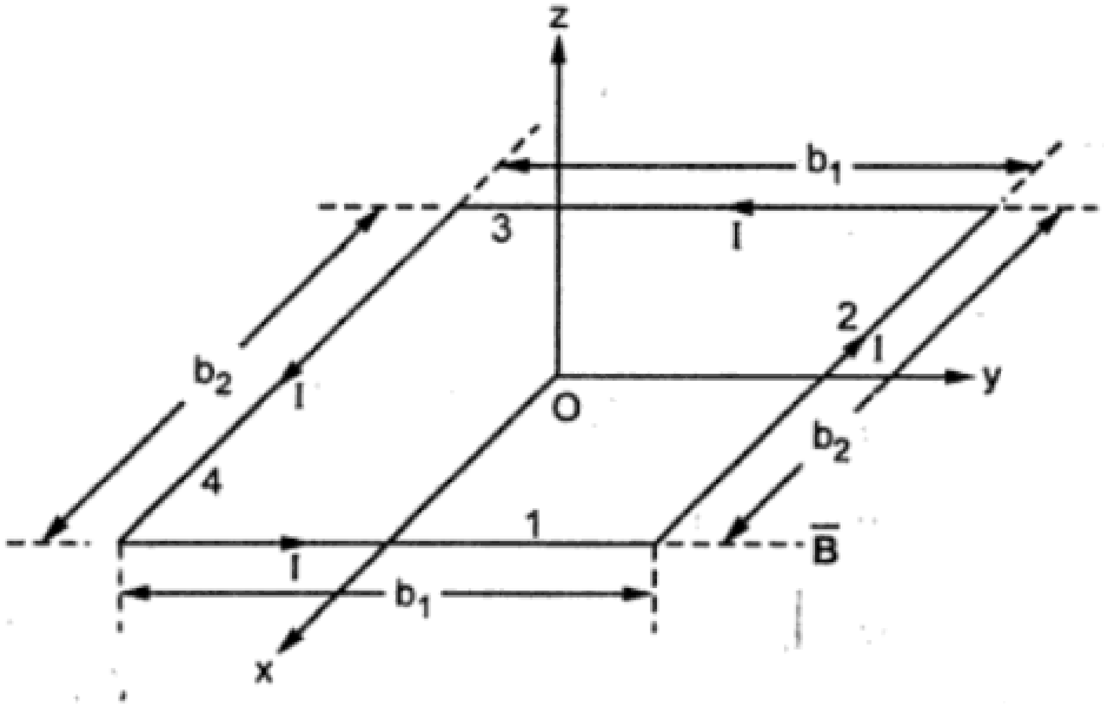
The vector force on the side 3 is given by,

$$\bar{F}_3 = I\, d\bar{L}_3 \times \bar{B}$$

$$= \left[I\left(-b_1 \bar{a}_y\right) \times \left(B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z\right)\right]$$

$$= - I b_1 \left[- B_x \bar{a}_z + B_z \bar{a}_x\right]$$

$$= I b_1 \left[- B_z \bar{a}_x + B_x \bar{a}_z\right]$$



Problem

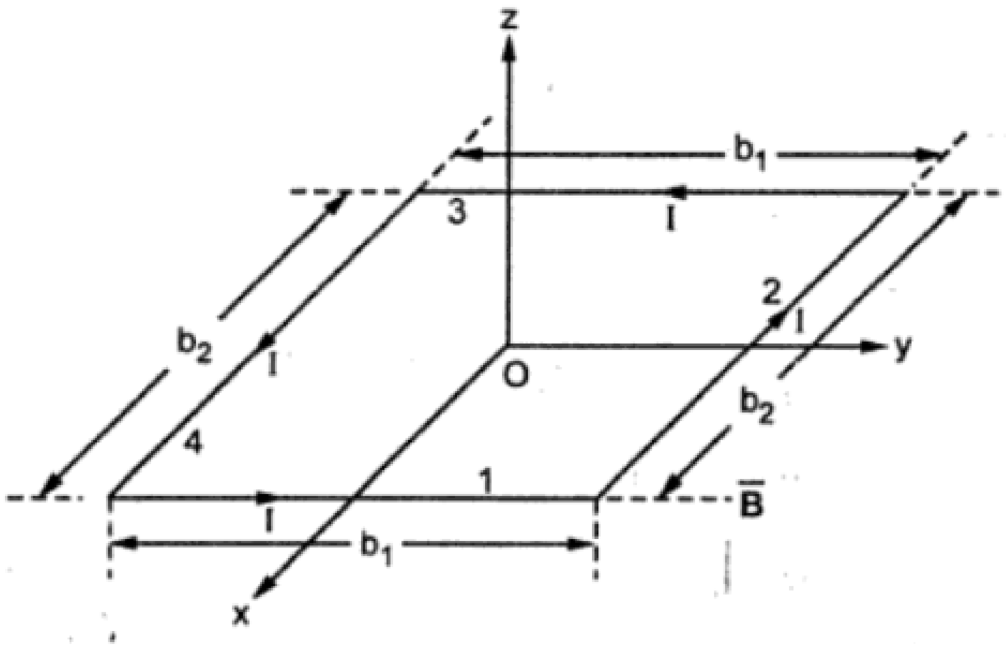
Finally the vector force on the side 4 is given by,

$$\bar{F}_4 = I\, d\bar{L}_4 \times \bar{B}$$

$$= \left[I\,(b_2\, \bar{a}_x) \times (B_x\, \bar{a}_x + B_y\, \bar{a}_y + B_z\, \bar{a}_z) \right]$$

$$= I\, b_2 \left[B_y\, \bar{a}_z - B_z\, \bar{a}_y \right]$$

$$= I\, b_2 \left[-B_z\, \bar{a}_y + B_y\, \bar{a}_z \right]$$



Hence total force on the loop of sides b_1 and b_2 is given by,

$$\bar{F} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$$

$$= I\, b_1 \left[B_z\, \bar{a}_x - B_x\, \bar{a}_z \right] - I\, b_2 \left[-B_z\, \bar{a}_y + B_y\, \bar{a}_z \right]$$

$$+ I\, b_1 \left[-B_z\, \bar{a}_x + B_x\, \bar{a}_z \right] + I\, b_2 \left[-B_z\, \bar{a}_y + B_y\, \bar{a}_z \right]$$

$$= I \left\{ \left(b_1\, B_z\, \bar{a}_x - b_1\, B_x\, \bar{a}_z - b_1\, B_z\, \bar{a}_x + b_1\, B_x\, \bar{a}_z \right) \right.$$

$$\left. + \left(b_2\, B_z\, \bar{a}_y - b_2\, B_y\, \bar{a}_z - b_2\, B_z\, \bar{a}_y + b_2\, B_y\, \bar{a}_z \right) \right\}$$

$$= 0$$

Problem

Total torque on the rectangular loop can be obtained by choosing origin of the torque at the centre of the loop. Hence total torque is given by.

$$\bar{T} = \bar{T}_1 + \bar{T}_2 + \bar{T}_3 + \bar{T}_4$$

But

$$\bar{T} = \bar{R}_1 \times \bar{F}_1 = \left(\frac{b_2}{2}\,\bar{a}_x\right) \times \left[I\, b_1\, (B_z\,\bar{a}_x - B_x\,\bar{a}_z)\right]$$

$$= I\left(\frac{b_1\, b_2}{2}\right)(B_x\,\bar{a}_y)$$

$$\bar{T}_2 = \bar{R}_2 \times \bar{F}_2 = \left(\frac{b_1}{2}\,\bar{a}_y\right) \times \left[-I\, b_2\, (-B_z\,\bar{a}_y + B_y\,\bar{a}_z)\right]$$

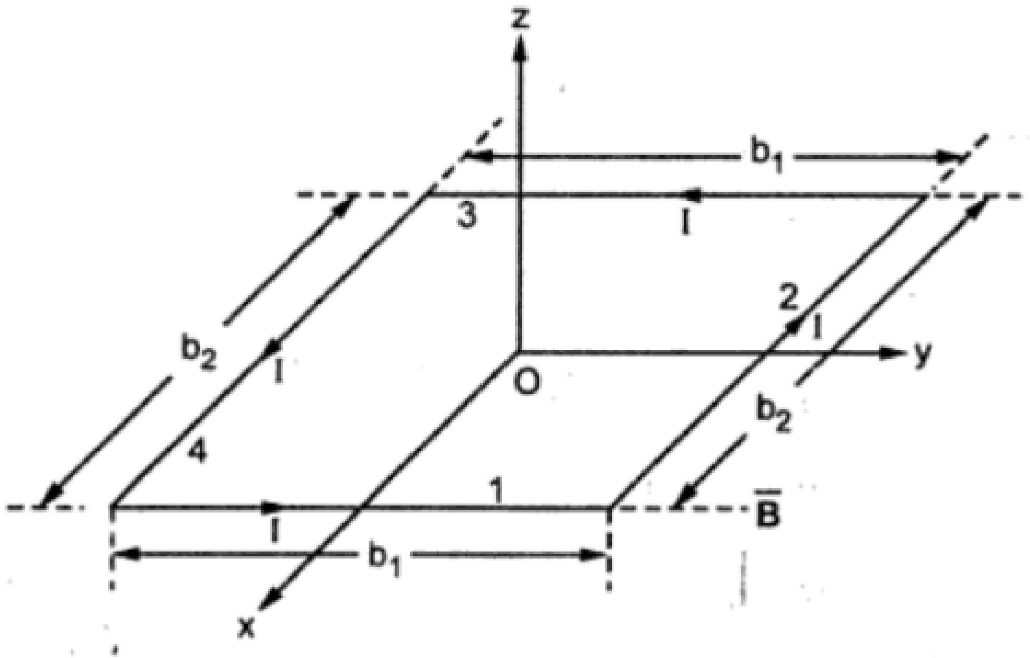
$$= -I\left(\frac{b_1\, b_2}{2}\right)(B_y\,\bar{a}_x)$$

$$= I\left(\frac{b_1\, b_2}{2}\right)(-B_y\,\bar{a}_x)$$

$$\bar{T}_3 = \bar{R}_3 \times \bar{F}_3 = \left(-\frac{b_2}{2}\,\bar{a}_x\right) \times \left[I\, b_1\, (-B_z\,\bar{a}_x + B_x\,\bar{a}_z)\right]$$

$$= -I\left(\frac{b_1\, b_2}{2}\right)(-B_x\,\bar{a}_y)$$

$$= I\left(\frac{b_1\, b_2}{2}\right)(B_x\,\bar{a}_y)$$

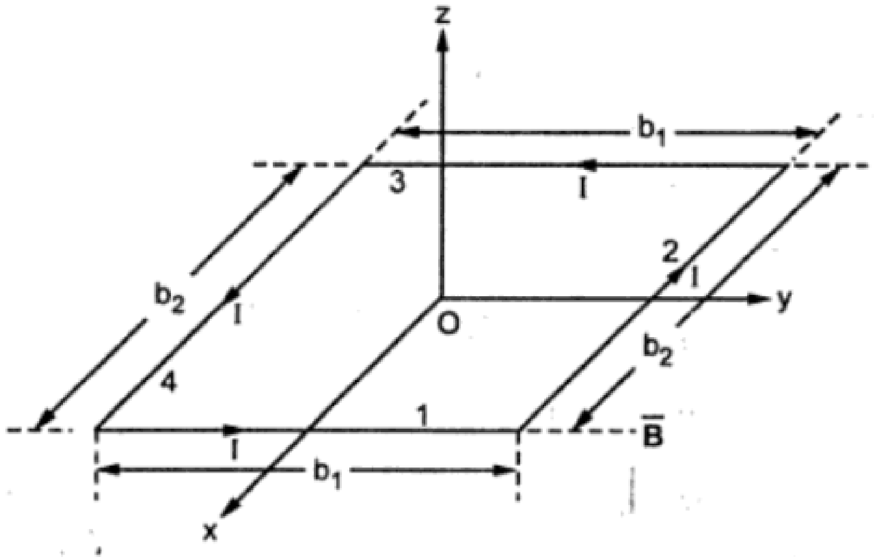


Problem

$$\bar{T}_4 = \bar{R}_4 \times \bar{F}_4 = \left(-\frac{b_1}{2}\,\bar{a}_y\right) \times \left[I\, b_2\left(-B_z\,\bar{a}_y + B_y\,\bar{a}_z\right)\right]$$

$$= -I\left(\frac{b_1\ b_2}{2}\right)\left(B_y\,\bar{a}_x\right)$$

$$= I\left(\frac{b_1\ b_2}{2}\right)\left(-B_y\,\bar{a}_x\right)$$



$$\bar{T} = \bar{T}_1 + \bar{T}_2 + \bar{T}_3 + \bar{T}_4$$

$$= I\left(\frac{b_1\ b_2}{2}\right)\left(B_x\,\bar{a}_y\right) + I\left(\frac{b_1\ b_2}{2}\right)\left(-B_y\,\bar{a}_x\right)$$

$$+ I\left(\frac{-b_1\ b_2}{2}\right)\left(B_x\,\bar{a}_y\right) + I\left(\frac{b_1\ b_2}{2}\right)\left(-B_y\,\bar{a}_x\right)$$

$$= I\left(\frac{b_1 b_2}{2}\right)\left[-2B_y\,\bar{a}_x + 2B_x\bar{a}_y\right]$$

$$= I\ b_1\ b_2\left(-B_y\,\bar{a}_x + B_x\,\bar{a}_y\right)$$

Magnetic dipole moment

The **magnetic dipole moment** of a current loop is defined as the product of current through the loop and the area of the loop, directed normal to the current loop. From the definition it is clear that, the magnetic dipole moment is a vector quantity. It is denoted by $\bar{m}$. The direction of the magnetic dipole moment $\bar{m}$ is given by the right hand thumb rule. The right hand thumb indicates the direction of the unit vector in which $\bar{m}$ is directed and the figures represents the current direction. The **magnetic dipole moment** is given by

$$\boxed{\bar{m} = (IS)\bar{a}_n \quad A \cdot m^2}$$

the expression for the torque along the axis of rotation of a planar coil as,

$$\bar{T} = BIS\left(-\bar{a}_y\right)$$

Using definition for the magnetic dipole moment, the torque can be expressed as,

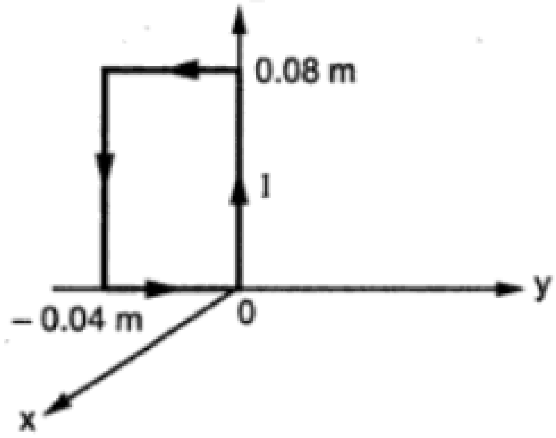
$$\boxed{\bar{T} = \bar{m} \times \bar{B} \quad Nm}$$

Problem

A rectangular coil is in the magnetic field given by,

$$\bar{B} = 0.05\,\frac{\bar{a}_x + \bar{a}_y}{\sqrt{2}}\ T$$

Find the torque about z-axis when the coil is in the position shown and carries a current of 5 A.



Solution : The magnetic dipole moment is given by,

$$\bar{m} = I\,S\,\bar{a}_n$$

Where S = Area of coil = $(0.08) \times (0.04) = 3.2 \times 10^{-3}\ m^2$

and $\bar{a}_n$ = Unit vector normal to the plane consisting a rectangular coil.

it is clear that the coil is placed in y-z plane. Hence a unit vector normal to y-z plane is in x-direction i.e. $\bar{a}_x$.

$$\therefore \quad \bar{m} = (5)\,(3.2 \times 10^{-3})\,\bar{a}_x$$

$$= 0.016\,\bar{a}_x\ A.m^2$$

Hence magnetic torque is given by,

$$\bar{T} = \bar{m} \times \bar{B} = (0.016\,\bar{a}_x) \times \left(0.05\,\frac{\bar{a}_x + \bar{a}_y}{\sqrt{2}}\right)$$

$$\therefore \quad \bar{T} = 5.6568 \times 10^{-4}\,[\bar{a}_x \times \bar{a}_x + \bar{a}_x \times \bar{a}_y]$$

$$\therefore \quad \bar{T} = 5.6568 \times 10^{-4}\ \bar{a}_z\ N.m \qquad \ldots \because\ \bar{a}_x \times \bar{a}_x = 0,\ \bar{a}_x \times \bar{a}_y = \bar{a}_z$$

Problem

A point charge, $Q = -60$ nC, is moving with a velocity 6×10^6 m/s in the direction specified by unit vector $-0.48\,\bar{a}_x - 0.6\,\bar{a}_y + 0.64\,\bar{a}_z$. Find the magnitude of the force on a moving charge in the magnetic field, $\bar{B} = 2\,\bar{a}_x - 6\,\bar{a}_y + 5\,\bar{a}_z$ mT.

Solution : The magnitude of velocity is given as $v = 6\times10^6$ m/s. The direction of this velocity is specified by an unit vector. Thus we can write,

$$\bar{v} = v\bar{a}_v = 6\times10^6\,[-0.48\bar{a}_x - 0.6\,\bar{a}_y + 0.64\,\bar{a}_z]\ \text{m/s}$$

The force experience by a moving charge in a steady magnetic field $\bar{B}$ is given by,

$$\bar{F} = Q\,\bar{v}\times\bar{B}$$

$$= -60\times10^{-9}\,[(6\times10^6)(-0.48\bar{a}_x - 0.6\bar{a}_y + 0.64\bar{a}_z)\times(2\bar{a}_x - 6\bar{a}_y + 5\,\bar{a}_z)(1\times10^{-3})]$$

$$= (-3.6\times10^{-4})\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ -0.48 & -0.6 & 0.64 \\ 2 & -6 & 5 \end{vmatrix}$$

$$= (-3.6\times10^{-4})[0.84\bar{a}_x + 3.68\,\bar{a}_y + 4.08\,\bar{a}_z]$$

$$= (-0.3024\,\bar{a}_x - 1.3248\,\bar{a}_y - 1.4688\,\bar{a}_z)\times10^{-3}\ \text{N}$$

Thus the magnitude of the force on a moving charge is given by,

$$|\bar{F}| = \sqrt{(-0.3024\times10^{-3})^2 + (-1.3248\times10^{-3})^2 + (-1.4688\times10^{-3})^2}$$

$$= \mathbf{2.0009\ mN}$$

Problem

A conductor of length 2.5 m in z = 0 and x = 4 m carries a current of 12 A in $-\bar{a}_y$ direction. Calculate the uniform flux density in the region, if the force on the conductor is 12×10^{-2} N in the direction specified by $\left[\frac{-\bar{a}_x+\bar{a}_z}{\sqrt{2}}\right]$.

Solution : Let $\bar{B} = B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_x$ T

The force exerted on the conductor is given by,

$$\bar{F} = I\,d\bar{L} \times \bar{B}$$

$$\therefore\ 12\times10^{-2}\left[\frac{-\bar{a}_x+\bar{a}_z}{\sqrt{2}}\right] = (-12\bar{a}_y)(2.5)\times(B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z)$$

$$\therefore\ -(0.0848)\,\bar{a}_x + (0.0848)\,\bar{a}_z = -30\,[\bar{a}_y \times (B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z)]$$

$$\therefore\ (2.8267\times10^{-3})\,\bar{a}_x - (2.8267\times10^{-3})\,\bar{a}_z = -B_x\,\bar{a}_z + B_z\,\bar{a}_x$$

Comparing components on both the sides of the equation,

$$B_x = 2.8267\times10^{-3}$$

$$B_y = 0$$

$$B_z = 2.8267\times10^{-3}$$

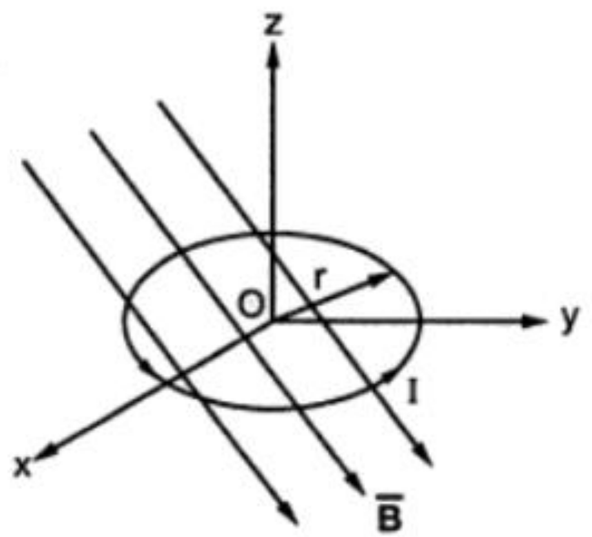
$$\bar{B} = 2.8267\times10^{-3}\,\bar{a}_x + 2.8267\times10^{-3}\,\bar{a}_z \text{ T}$$

$$\bar{B} = (2.8267\,\bar{a}_x + 2.8267\,\bar{a}_z) \text{ mT}$$

Problem

A circular loop of radius r and current I lies in z = 0 plane. Find the torque which results if the current is in $\bar{a}_\phi$ and there is a uniform field $\bar{B} = \frac{B_0}{\sqrt{2}}(\bar{a}_x + \bar{a}_z)$ T.

Solution : Consider a circular loop in z = 0 plane



Current is in $\bar{a}_\phi$ as shown in the Fig. 8.26. The given magnetic field is uniform given by,

$$\bar{B} = B_0\left(\frac{\bar{a}_x + \bar{a}_z}{\sqrt{2}}\right) \text{ T}$$

The magnetic dipole moment of a planar circular loop is given by,

$$\bar{m} = (I\,S)\;\bar{a}_n$$

where S is the area of the circular loop.

Note that the loop is laying in z = 0 plane. Thus the direction of unit normal $\bar{a}_n$ must be decided by the right hand thumb rule. Let the fingures point in the direction of current (in $\bar{a}_\phi$ direction), then the right thumb gives the direction of $\bar{a}_n$ which is clearly $\bar{a}_z$.

$$\therefore \qquad \bar{m} = I(\pi r^2)\,\bar{a}_z = (\pi r^2 I)\,\bar{a}_z$$

Problem

The total torque is given by,

$$\bar{T} = \bar{m} \times \bar{B}$$

$$= (\pi r^2 I)\,\bar{a}_z \times \frac{B_0}{\sqrt{2}}(\bar{a}_x + \bar{a}_z)$$

$$= \frac{\pi r^2 B_0 I}{\sqrt{2}}[\bar{a}_z \times (\bar{a}_x + \bar{a}_z)]$$

$$= \frac{\pi r^2 B_0 I}{\sqrt{2}}\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \frac{\pi r^2 B_0 I}{\sqrt{2}}[-(-\bar{a}_y)]$$

$$= \left(\frac{\pi r^2 B_0 I}{\sqrt{2}}\right)\bar{a}_y \quad \text{N.m}$$

Faraday's Law

In year 1820, **Prof. Hans Christian Oersted** demonstrated that a compass needle deflected due to an electric current. After ten years, **Michael Faraday**, a British Scientist, proved that a magnetic field could produce a current.

According to Faraday's experiment, a static magnetic field cannot produce any current flow. But with a time varying field, an electromotive force (e.m.f.) induces which may drive a current in a closed path or circuit. This e.m.f. is nothing but a voltage that induces from changing magnetic fields or motion of the conductors in a magnetic field. Faraday discovered that the induced e.m.f. is equal to the time rate of change of magnetic flux linking with the closed circuit.

Faraday's law can be stated as,

$$\boxed{e = -N\frac{d\phi}{dt} \text{ volts.}}$$

where N = Number of turns in the circuit

e = Induced e.m.f.

Faraday's Law

the induced e.m.f. is given by,

$$e = \oint \bar{E} \cdot d\bar{L}$$

The induced e.m.f. indicates a voltage about a closed path such that if any part of the path is changed, the e.m.f. will also change.

The magnetic flux ϕ passing through a specified area is given by

$$\phi = \int_S \bar{B} \cdot d\bar{S}$$ where B = Magnetic flux density

$$e = -\frac{d}{dt}\int_S \bar{B} \cdot d\bar{S}$$

$$e = \oint \bar{E} \cdot d\bar{L} = -\frac{d}{dt}\int_S \bar{B} \cdot d\bar{S}$$

Faraday's Law

(i) The closed circuit in which e.m.f. is induced is stationary and the magnetic flux is sinusoidally varying with time. it is clear that the magnetic flux density is the only quantity varying with time. We can use partial derivative to define relationship as $\bar{\mathbf{B}}$ may be changing with the co-ordinates as well as time. Hence we can write,

$$\boxed{\oint \bar{\mathbf{E}} \bullet d\bar{\mathbf{L}} = -\int_S \frac{\partial \bar{\mathbf{B}}}{\partial t} \bullet d\bar{\mathbf{S}}}$$

This is similar to **transformer action** and e.m.f. is called **transformer e.m.f.**. Using Stoke's theorem, a line integral can be converted to the surface integral as

$$\oint_S (\nabla \times \bar{\mathbf{E}}) \bullet d\bar{\mathbf{S}} = -\int_S \frac{\partial \bar{\mathbf{B}}}{\partial t} \bullet d\bar{\mathbf{S}}$$

Assuming that both the surface integrals taken over identical surfaces.

$$\therefore \quad (\nabla \times \bar{\mathbf{E}}) \bullet d\bar{\mathbf{S}} = -\frac{\partial \bar{\mathbf{B}}}{\partial t} \bullet d\bar{\mathbf{S}}$$

$$\nabla \times \bar{\mathbf{E}} = -\frac{\partial \bar{\mathbf{B}}}{\partial t}$$

If $\bar{\mathbf{B}}$ is not varying with time,

$$\oint \bar{\mathbf{E}} \bullet d\bar{\mathbf{L}} = 0, \text{ and}$$

$$\boxed{\nabla \times \bar{\mathbf{E}} = 0}$$

ii) Secondly magnetic field is stationary, constant not varying with time while the closed circuit is revolved to get the relative motion between them. This action is similar to **generator action**, hence the induced e.m.f. is called **motional** or **generator e.m.f.**

Consider that a charge Q is moved in a magnetic field $\bar{\mathbf{B}}$ at a velocity $\bar{v}$. Then the force on a charge is given by,

$$\bar{\mathbf{F}} = Q\,\bar{v} \times \bar{\mathbf{B}}$$

But the motional electric field intensity is defined as the force per unit charge. It is given by,

$$\therefore \qquad \bar{\mathbf{E}}_m = \frac{\bar{\mathbf{F}}}{Q} = \bar{v} \times \bar{\mathbf{B}}$$

Thus the induced e.m.f. is given by

$$\boxed{\oint \bar{\mathbf{E}}_m \cdot d\bar{\mathbf{L}} = \oint \left(\bar{v} \times \bar{\mathbf{B}}\right) \cdot d\bar{\mathbf{L}}}$$

iii) If in case, the magnetic flux density is also varying with time, then the induced e.m.f. is the combination of transformer e.m.f. and generator e.m.f. given by,

$$\boxed{\oint \bar{E} \cdot d\bar{L} = -\int \frac{\partial \bar{B}}{\partial t} \cdot d\bar{S} + \oint (\bar{v} \times \bar{B}) \cdot d\bar{L}}$$

Faraday's Law

$$e = -\frac{d\phi}{dt} = \oint \bar{E} \bullet d\bar{L}$$

Conditions for induced emf	Expressions for induced emf
i) Loop stationary, but magnetic flux density varies with time	$\oint \bar{E} \bullet d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \bullet d\bar{S}$
ii) Magnetic field stationary but loop is moving	$\oint \bar{E} \bullet d\bar{L} = \oint (\bar{v} \times \bar{B}) \bullet d\bar{L}$
iii) Loop is moving and the magnetic field varying with time	$\oint \bar{E} \bullet d\bar{L} = \oint (\bar{v} \times \bar{B}) \bullet d\bar{L} - \int_S \frac{\partial \bar{B}}{\partial t} \bullet d\bar{S}$
Fundamental postulate for an electromagnetic induction	$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

Problem

A conductor 1 cm in length is parallel to z-axis and rotates at radius of 25 cm at 1200 rpm. Find induced voltage, if the radial field is given by

$$\bar{B} = 0.5\,\bar{a}_r \ T$$

Solution : In above case, the magnetic flux is constant while the path is rotating at 1200 rpm. Under such condition, the field intensity is given by,

$$\bar{E} = \bar{v} \times \bar{B}$$

where $\bar{v}$ = linear velocity

In 1 minute there are 1200 revolutions which corresponds to 20 revolutions in one second. In one revolution distance travelled is $(2\pi r)$ meter. Hence in 20 revolutions the distance travelled in one second is $(40\pi r)$ meter. The conductor rotates in ϕ - direction. Hence linear velocity is given by,

$$\bar{v} = (40\pi r)\bar{a}_\phi$$

$$= 40\pi(25\times 10^{-2})\bar{a}_\phi$$

$$= 31.416\,\bar{a}_\phi \text{ m/s}$$

Hence an electric field intensity is calculated as,

$$\bar{E} = [31.416\bar{a}_\phi] \times [0.5\bar{a}_r]$$

$$= 15.708\,(-\bar{a}_z) \qquad \ldots\ \bar{a}_\phi \times \bar{a}_r = -\bar{a}_z$$

Problem

Induced voltage is given by,

$$e = \oint \bar{E} \cdot d\bar{L}$$

Now $\quad d\bar{L} = (dz)\,\bar{a}_z \quad$ as conductor is parallel to z-axis.

$$e = \int_{z=0}^{0.01} 15.708(-\bar{a}_z) \cdot (dz)\,\bar{a}_z$$

$$= -15.708\,[z]_0^{0.01} = \mathbf{-157.08\ mV}$$

Negative sign indicates upper end of the conductor is positive while lower end is negative. Thus the magnitude of the induced voltage is 157.08 mV.

Problem

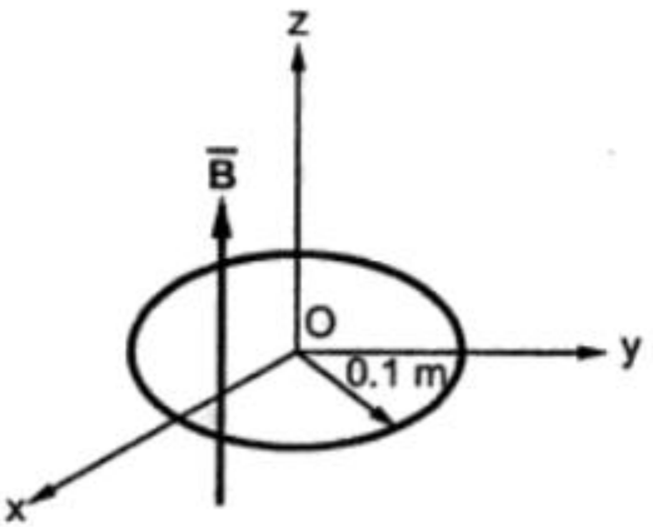
A circular loop conductor lies in plane z = 0 and has a radius of 0.1 m and resistance of 5 Ω. Given $\bar{B} = 0.2 \sin 10^3 t \, \bar{a}_z$ *J, determine the current in the loop.*

Solution : To find current in the loop, let us first calculate induced e.m.f.

A circular loop is in z = 0 plane. $\bar{B}$ is in z-direction which is perpendicular to the loop. So $\bar{B}$ is perpendicular to the circular loop.

Hence total flux is given by,

$$\Phi = \int_S \bar{B} \cdot d\bar{S}$$



With cylindrical co-ordinate system,

$$d\bar{S} = (r \, dr \, d\phi) \, \bar{a}_z$$

Problem

$$\Phi = \int_{\phi=0}^{2\pi} \int_{r=0}^{0.1} \left[\left(0.2 \sin 10^3\, t\right)\bar{a}_z\right] \bullet \left[(r\, dr\, d\phi)\,\bar{a}_z\right]$$

$$\Phi = \left(0.2 \sin 10^3\, t\right)[\phi]_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{0.1}$$

$$\Phi = \left(0.2 \sin 10^3\, t\right)[2\pi]\left[\frac{(0.1)^2}{2}\right]$$

$$\Phi = 6.283\times 10^{-3} \sin 10^3 t$$

$$\Phi = 6.283 \sin 10^3 t \quad \text{mWb}$$

induced e.m.f. is given by,

$$e = -\frac{d\Phi}{dt}$$

$$= -\frac{d}{dt}\left[6.283\times 10^{-3} \sin 10^3 t\right]$$

$$= -6.283\times 10^{-3} \times 10^3 \times \cos 10^3 t$$

$$= -6.283 \cos 10^3 t \quad \text{V}$$

Hence the current in the conductor is given by,

$$i = \frac{\text{Induced e.m.f.}}{\text{Resistance}}$$

$$\therefore \quad i = \frac{-6.283 \cos 10^3 t}{5}$$

$$\therefore \quad i = -1.2567 \cos 10^3 t \quad \text{A}$$

Displacement Current

For static electromagnetic fields, according to Ampere's circuital law, we can write,

$$\nabla \times \overline{H} = \overline{J} \quad \text{... (1)}$$

Taking divergence on both the sides,

$$\nabla \bullet (\nabla \times \overline{H}) = \nabla \bullet \overline{J}$$

But according to vector identity, **'divergence of the curl of any vector field is zero'**. Hence We can write,

$$\nabla \bullet (\nabla \times \overline{H}) = \nabla \bullet \overline{J} = 0 \quad \text{... (2)}$$

But the equation of continuity is given by,

$$\nabla \bullet \overline{J} = -\frac{\partial \rho_v}{\partial t} \quad \text{... (3)}$$

From equation (3) it is clear that when $\frac{\partial \rho_v}{\partial t} = 0$, then only equation (2) becomes true. Thus equations (2) and (3) are not compatible for time varying fields. We must modify equation (1) by adding one unknown term say $\overline{N}$.

Then equation (1) becomes,

$$\nabla \times \overline{H} = \overline{J} + \overline{N} \quad \text{... (4)}$$

Displacement Current

Again taking divergence on both the sides

$$\nabla \bullet (\nabla \times \bar{H}) = \nabla \bullet \bar{J} + \nabla \bullet \bar{N} = 0$$

As $\nabla \bullet \bar{J} = -\frac{\partial \rho_v}{\partial t}$, to get correct conditions we must write,

$$\nabla \bullet \bar{N} = \frac{\partial \rho_v}{\partial t}$$

But according to Gauss's law,

$$\rho_v = \nabla \bullet \bar{D}$$

Thus replacing ρ_v by $\nabla \bullet \bar{D}$

$$\nabla \bullet \bar{N} = \frac{\partial}{\partial t}(\nabla \bullet \bar{D})$$

$$= \nabla \bullet \frac{\partial \bar{D}}{\partial t}$$

Comparing two sides of the equation,

$$\bar{N} = \frac{\partial \bar{D}}{\partial t}$$

Displacement Current

Now we can write Ampere's circuital law in point form as,

$$\boxed{\nabla \times \bar{H} = \bar{J}_C + \frac{\partial \bar{D}}{\partial t}} \qquad \text{.... (6)}$$

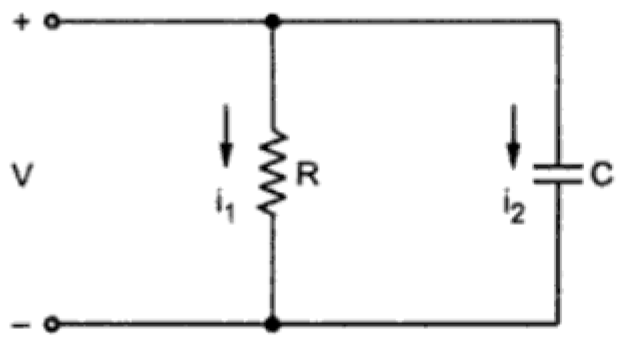
The first term in equation (6) is **conduction current density** denoted by $\bar{J}_C$. Here attaching subscript C indicates that the current is due to the moving charges.

The second term in equation (6) represents current density expressed in ampere per square meter. As this quantity is obtained from time varying electric flux density. This is also called **displacement density**. Thus this is called **displacement current density** denoted by $\bar{J}_D$. With these definitions we can write equation (6) as,

$$\boxed{\nabla \times \bar{H} = \bar{J}_C + \bar{J}_D} \qquad \text{... (7)}$$

Displacement Current

Consider a parallel circuit of a resistor and capacitor driven by a time varying voltage V



Let the current flowing through resistor R be i_1 and the current flowing through capacitor C be i_2. The nature of the current flowing through the resistor R is different than that flowing through the capacitor. The current through resistor is due to the actual motion of charges. Thus the current through resistor can be written as,

$$i_1 = \frac{V}{R} \quad \dots (8)$$

This current is called **conduction current** as the current is flowing because of actual motion of charges. Let it be denoted by i_C.

Let A be the cross-sectional area of resistor, then the conduction current density is given by,

$$\boxed{\bar{J}_C = \frac{i_C}{A} = \sigma \bar{E}} \quad \dots (9)$$

Displacement Current

Now assume that the initial charge on a capacitor is zero. Then for time varying voltage applied across parallel plate capacitor, the current through the capacitor is given by,

$$i_2 = C\frac{dv}{dt}$$

Let the two plates of area A are separated by distance d with dielectric having permittivity ε in between the plates. Then we can write

$$i_2 = \frac{\varepsilon A}{d}\frac{dv}{dt}$$

Now this current is called **displacement current** denoted by i_D. The electric field produced by the voltage applied between the two plates is given by,

$$E = \frac{V}{d}$$

or

$$V = (d)(E)$$

$$i_D = i_2 = \frac{\varepsilon A}{d}\frac{d}{dt}(dE)$$

$$i_D = \frac{\varepsilon A}{d} d \frac{dE}{dt} \qquad \text{... As distance d is not varying with time}$$

$\therefore$

Displacement Current

Now the ratio of current to the area of plate is the current density. In this case it is displacement current density denoted by J_D.

$$\therefore \quad J_D = \frac{i_D}{A}$$

$$\therefore \quad J_D = \frac{d}{dt}(\varepsilon \bar{E})$$

$$\therefore \quad J_D = \frac{\partial \bar{D}}{\partial t}$$

Thus in a given medium, both the types of the currents, namely the conduction current current and the displacement current may flow. Hence the two current densities can be written as,

$$\boxed{\begin{array}{ll} \bar{J}_C = \bar{E} & \text{....conduction current density} \\ \bar{J}_D = \dfrac{\partial \bar{D}}{\partial t} & \text{....displacement current density} \end{array}}$$

The total current density is given by,

$$J = J_C + J_D$$

Displacement Current

Some materials are good conductors while some are perfect dielectrics. But in some materials which are neither good conductor nor perfect dielectrics, both the current namely conduction current and displacement current may exist.

For the electric field intensity $\bar{E}$, let the time dependence be given by $e^{j\omega t}$, the total current density is given by,

$$J = J_C + J_D$$

$$= \sigma \bar{E} + \frac{\partial \bar{D}}{\partial t}$$

$$= \sigma \bar{E} + \frac{\partial}{\partial t}(\varepsilon \bar{E})$$

$$\therefore \quad \boxed{\bar{J} = \sigma \bar{E} + j\omega\varepsilon \bar{E}}$$

Then the ratio of the magnitudes of the conduction current density to the displacement current density is given by,

$$\boxed{\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon}}$$

Thus the ratio of the magnitudes of the conduction current density to the displacement current density depends on the properties of the medium (i.e. σ and μ) and the frequency (i.e. ω). For a conductor, the value of conductivity σ is very large.

Key Point : *So in conductor, the conduction current is very large as compared to the displacement current.*

While for a dielectric, the value of conductivity σ is very small.

Key Point : *So in dielectric medium, the displacement current is greater as compared to the conduction current.*

In other words, if the ratio of the magnitudes of the current densities is greater than 1, the medium is conductor and if the ratio of the magnitudes is less than 1 then the medium is dielectric

If	$\frac{\sigma}{\omega\varepsilon} >> 1,$	medium is conductor
If	$\frac{\sigma}{\omega\varepsilon} << 1,$	medium is dielectric

Also the ratio represented above depends on frequency, a medium which is conductor at low frequency may become insulator at very high frequency.

Displacement Current

- According to Faraday's law of electromagnetic induction, a time-varying magnetic field induces an emf,
- According to Maxwell, an electric field sets up a current and hence a magnetic field. Such a current is called displacement current.
- The current that exists inside the capacitor is Displacement current.
- Conduction current is due to the flow of electrons in a circuit. It exists even if electrons flow at a uniform rate.
- Displacement current is due to the time-varying electric field. It does not exist under steady conduction.
- The magnitude of displacement current in case of steady electric fields in a conducting wire is zero, since the electric field E does not change with time.

Displacement Current

- When a capacitor starts charging there is no conduction of charge between the plates.
  However, because of change in charge accumulation with time above the plates,
  the electric field changes causing the displacement current.
- Conduction current is the actual current whereas displacement current is the apparent current produced by time varying electric field.
- The current due to the changing electric field is called displacement current.
- The displacement current satisfy the property of continuity i.e the sum of displacement and conduction current remains constant along the closed path.
- The magnitude of displacement current in case of steady electric fields in a conducting wire is Zero.

Problem

In a given lossy dielectric medium, conduction current density $J_C = 0.02 \sin 10^9 t\,(A/m^2)$. *Find the displacement current density if* $\sigma = 10^3$ *S/m and* $\varepsilon_r = 6.5$.

Solution : For lossy dielectric medium,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon}$$

$$\therefore \quad J_D = \frac{\omega\varepsilon J_C}{\sigma} = \frac{10^9 \times (\varepsilon_r \varepsilon_0) \times 0.02}{10^3}$$

$$\therefore \quad J_D = \frac{10^9 \times 6.5 \times 8.854 \times 10^{-12} \times 0.02}{10^3}$$

$$\therefore \quad J_D = 1.151 \times 10^{-6}\ A/m^2 = \mathbf{1.151\ \mu A/m^2}$$

As $\bar{J}_D$ and $\bar{J}_C$ are always at right angles to each other, we can write,

$$\bar{J}_D = \mathbf{1.151 \cos 10^9 t\ \mu A/m^2}$$

Maxwells Equation

Differential form	Integral form	Significance
$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint \bar{E} \bullet d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \bullet d\bar{S}$	Faraday's law
$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$	$\oint \bar{H} \bullet d\bar{L} = I + \int_S \frac{\partial \bar{D}}{\partial t} \bullet d\bar{S}$	Ampere's circuital law
$\nabla \bullet \bar{D} = \rho_v$	$\oint_S \bar{D} \bullet d\bar{S} = \int_S \rho_v \, dv$	Gauss's law
$\nabla \bullet \bar{B} = 0$	$\oint_S \bar{B} \bullet d\bar{S} = 0$	No isolated magnetic charges.

Problem

$$\text{If } \overline{D} = 10x\bar{a}_x - 4y\bar{a}_y + kz\bar{a}_z \ \ \mu C/m^2 \text{ and } \overline{B} = 2\bar{a}_y \ mT,$$

Find the value of k to satisfy the Maxwell's equations for region $\sigma = 0$, $\rho_v = 0$.

Solution : As $\sigma = 0$ and $\rho_v = 0$, the medium in which $\overline{D}$ and $\overline{B}$ are present is nothing but free space. So the Maxwell's equation obtained from Gauss's law is given by

$$\nabla \cdot \overline{D} = \rho_v = 0 \quad \text{... for free space.}$$

$$\therefore \quad \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0$$

$$\therefore \quad \frac{\partial}{\partial x}(10x) + \frac{\partial}{\partial y}(-4y) + \frac{\partial}{\partial z}(kz) = 0$$

$$\therefore \quad 10 - 4 + k = 0$$

$$\therefore \quad k = -6 \ \mu C/m^3$$

Problem

If the magnetic field $\bar{H} = [3x\cos\beta + 6y\sin\alpha]\bar{a}_z$, *find current density* $\bar{J}$ *if fields are invariant with time.*

Solution : The point form of Maxwell's second equation is

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

But as fields are time invariant, we can write,

$$\frac{\partial \bar{D}}{\partial t} = 0$$

$$\therefore \quad \nabla \times \bar{H} = \bar{J}$$

$$\therefore \quad \bar{J} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & (3x\cos\beta + 6y\sin\alpha) \end{vmatrix}$$

$$\therefore \quad \bar{J} = \frac{\partial}{\partial y}[3x\cos\beta + 6y\sin\alpha]\,\bar{a}_x - \frac{\partial}{\partial x}[3x\cos\beta + 6y\sin\alpha]\,\bar{a}_y$$

$$\therefore \quad \bar{J} = 6\sin\alpha\,\bar{a}_x - 3\cos\beta\,\bar{a}_y \quad A/m^2$$

Problem

Find the frequency at which conduction current density and displacement current density are equal in a medium with $\sigma = 2\times10^{-4}$ *℧/m and* $\varepsilon_r = 81$.

Solution : The ratio of amplitudes of the two current densities is given as 1, so we can write,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon} = 1$$

i.e. $$\omega = \frac{\sigma}{\varepsilon} = \frac{\sigma}{\varepsilon_0\,\varepsilon_r}$$

$\therefore$ $$\omega = \frac{2\times10^{-4}}{\left(8.854\times10^{-12}\right)(81)} = 0.2788\times10^{6} \text{ rad/sec}$$

But $$\omega = 2\pi f$$

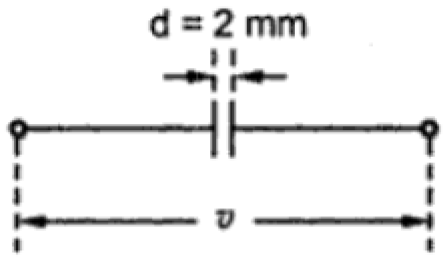
$\therefore$ $$f = \frac{\omega}{2\pi} = \frac{0.2788\times10^{6}}{2\pi} = \mathbf{44.372 \text{ kHz}}$$

Hence, the frequency at which the ratio of amplitudes of conduction and displacement current density is unity, is 44.372 kHz.

Problem

Two parallel conducting plates of area 0.05 m^2 are separated by 2 mm of a lossy dielectric for which $\varepsilon_r = 8.3$ and $\sigma = 8\times10^{-4}$ S/m. Given an applied voltage $v = 10\sin 10^7 t$ V. Find total r.m.s. current.

The electric field produced due to the applied voltage v is given by,



$$E = \frac{v}{d} = \frac{10\sin 10^7 t}{2\times10^{-3}} = 5000\sin 10^7 t \ \text{V/m}$$

$$J_C = \sigma E = \left(8\times10^{-4}\right)\left(5000\sin 10^7 t\right) = 4\sin 10^7 t \ \text{A/m}^2$$

$$J_D = \varepsilon\frac{dE}{dt} = \varepsilon_0\varepsilon_r\frac{dE}{dt}$$

$$= 8.854\times10^{-12}\times8.3\frac{d}{dt}\left[5000\sin 10^7 t\right]$$

$$= 8.854\times10^{-12}\times8.3\times5000\times10^7\times\cos 10^7 t$$

$$= 3.6744\cos 10^7 t \ \text{A/m}^2$$

Problem

The conduction current i_C is given by

$$i_C = (J_C)(\text{Area}) = \left(4 \sin 10^7 \; t\right)(0.05) = 0.2 \sin 10^7 \; t \;\; A$$

The displacement current i_D is given by

$$i_D = (J_D)(\text{Area}) = \left(3.6744 \cos 10^7 \; t\right)(0.05)$$

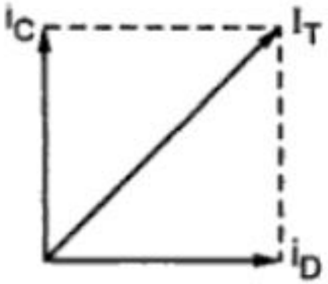
$$= 0.18372 \cos 10^7 t \; A$$

Both the currents are at right angles to each other

$$I_T = \sqrt{i_C^2 + i_D^2}$$

$$= \sqrt{(0.2)^2 + (0.1837)^2}$$

$$= 0.2715 \; A$$



Hence total r.m.s. current is given by

$$I_{T(r.m.s.)} = \frac{I_T}{\sqrt{2}} = \frac{0.2715}{\sqrt{2}} = \mathbf{0.1919 \; A}$$

Problem

Find the displacement current density within a parallel plate capacitor having a dielectric with $\varepsilon_r = 10$, area of plates $A = 0.01\ m^2$, distance of separation $d = 0.05$ mm. Applied voltage is $V = 200 \sin 200\ t$.

Solution : Current through a parallel plate capacitor is given by,

$$i_C = \left(\frac{\varepsilon \cdot A}{d}\right)\frac{dV}{dt} = \left(\frac{\varepsilon_0 \varepsilon_r \cdot A}{d}\right)\frac{dV}{dt}$$

Putting values of $\varepsilon_0, \varepsilon_r$, A, d and V,

$$i_C = \frac{\left(8.854\times10^{-12}\right)(10)(0.01)}{0.05\times10^{-3}} \cdot \frac{d}{dt}[200 \sin 200\ t]$$

$$\therefore \qquad i_C = 0.7083 \times 10^{-3} \cos 200\ t \ \ A$$

As we know for parallel plate capacitor,

$$i_C = i_D,$$

The displacement current density is given by

$$J_D = \frac{\text{Current}}{\text{Area}} = \frac{i_D}{A} = \frac{0.7083\times10^{-3}\ \cos 200\ t}{0.01}$$

$$\therefore \qquad J_D = 70.832 \times 10^{-3} \cos 200\ t \ \ A/m^2$$

Problem

Find the induced voltage in the conductor if $\bar{B} = 0.04\,\bar{a}_y T$ and $\bar{v} = 2.5 \sin 10^3\, t\, \bar{a}_z$ m/s.

Find inducted e.m.f. if $\bar{B}$ is changed to $0.04\,\bar{a}_x$ T.

Solution : (a) The induced e.m.f is given by,

$$e = \oint \bar{E} \cdot d\bar{L}$$

$$\text{But } \oint \bar{E} \cdot d\bar{L} = \oint (\bar{v} \times \bar{B}) \cdot d\bar{L}$$

$$\therefore \quad e = \int_0^{0.2} [2.5 \sin 10^3\, t\, \bar{a}_z \times 0.04\, \bar{a}_y] \cdot [dx\, \bar{a}_x]$$

$$\therefore \quad e = \int_0^{0.2} [0.1 \sin 10^3\, t\, (-\bar{a}_x)] \cdot [dx\, \bar{a}_x]$$

$$\therefore \quad e = -0.1 \sin 10^3\, t \int_0^{0.2} dx$$

$$\therefore \quad e = -0.1 \sin 10^3\, t\, [x]_0^{0.2}$$

$$\therefore \quad e = -0.1 \sin 10^3\, t\, [0.2]$$

$$\therefore \quad e = -\ \mathbf{0.02 \sin 10^3\, t\ V}$$

(b) If $\bar{B}$ is changed to $\bar{B} = 0.04\, \bar{a}_x$ T then the conductor cannot cut field lines hence induced voltage will be zero.

Problem : GATE 2015

Match the following

P	Stokes's Theorem	1	$\oint\!\!\oint D \cdot ds = Q$
Q	Gauss's Theorem	2	$\oint f(z)dx = 0$
R	Divergence Theorem	3	$\iiint (\nabla \cdot A)dv = \oint\!\!\oint A.ds$
S	Cauchy's Integral Theorem	4	$\iint (\nabla \times A).ds = \oint A.dl$

(A) P-2, Q-1, R-4, S-3

(B) P-4, Q-1, R-3, S-2

(C) P-4, Q-3, R-1, S-2

(D) P-3, Q-4, R-2, S-1

Correct option is (B).

Problem : GATE 2000

An electron with velocity, $\vec{u}$ is placed in an electric field, $\vec{E}$ and magnetic field, $\vec{B}$. The force experienced by the electron is given by

(a) $-e\vec{E}$

(b) $e\vec{u}\times\vec{B}$

(c) $-e(\vec{u}\times\vec{E}+\vec{B})$

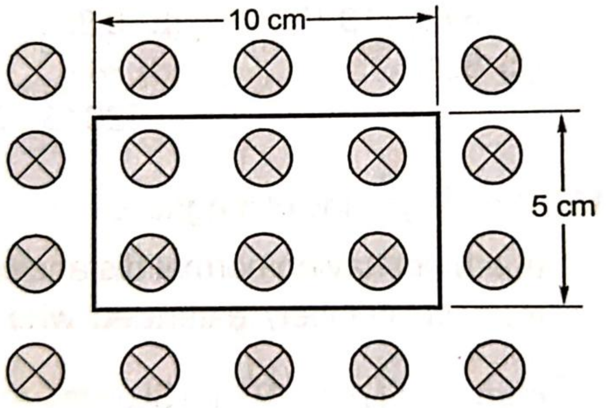
(d) $-e(\vec{E}+\vec{u}\times\vec{B})$

[2000 : 1 Mark]

Option (d) is Correct

Problem : GATE 2015

Consider a one-turn rectangular loop of wire placed in a uniform magnetic field as shown in the figure. The plane of the loop is perpendicular to the field lines. The resistance of the loop is 0.4 Ω, and its inductance is negligible. The magnetic flux density (in Tesla) is a function of time, and is given by $B(t) = 0.25\sin\omega t$, where $\omega = 2\pi \times 50$ radian/second. The power absorbed (in Watt) by the loop from the magnetic field is ______.



[2015 : 1 Mark, Set-1]

$$A = 10\text{ cm} \times 5\text{ cm};$$

$$B(t) = 0.25\sin\omega t,\ R = 0.4\ \Omega$$

$$P = i^2R$$

$$\because \quad e = -\frac{d\phi}{dt} = -\frac{d}{dt}(B\times A)$$

$$= -\frac{d}{dt}[0.25\times 50\times 10^{-4}\sin\omega t]$$

$$e = -0.25\times 50\times 10^{-4}\,\omega\cos\omega t$$

$$P_{avg.} = \frac{e_{rms}^2}{R}$$

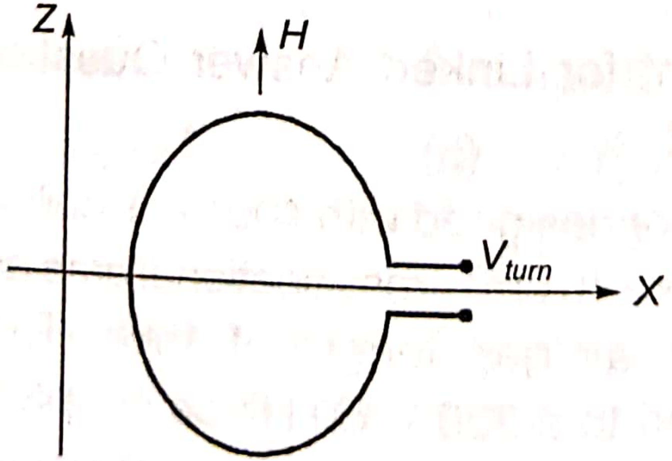
$$P = \int_0^T \frac{(12.5\times 100\pi\times 10^{-4})^2}{0.4}\times\cos^2\omega t\, d(\omega t)$$

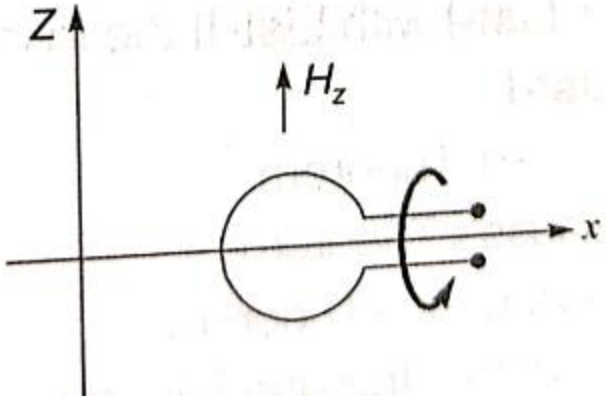
$$= (12.5\times\pi\times 10^{-2})^2\times\frac{1}{2}\times 0.4$$

$$= 0.193\text{ Watt}$$

Problem : GATE 2015

A circular turn of radius 1 m revolves at 60 rpm about its diameter aligned with the x-axis as shown in the figure. The value of μ_0 is $4\pi \times 10^{-7}$ in SI unit. If a uniform magnetic field intensity $\vec{H} = 10^7 \hat{z}$ A/m is applied, them the peak value if the induced voltage, V_{turn} (in Volts), is ________.





[2015 : 1 Mark, Set-2]

The circular turn rotate with 60 rpm, let the angle made by ring w.r.t. x-axis θ

and $\theta = \omega_0 t$, the turn rotate at 60 rpm,

so $\omega_0 = 2\pi$

so, the flux flowing through the circular turn will be

$\Psi = (\mu_0 H_z \times \text{Area of turn} \times \cos\omega_0 t)$

$\Psi = 4\pi \times 10^{-7} \times 10^7 \text{ A/m} \times \pi \times 1^2 \times \cos\omega_0 t$

Maximum voltage induced is $\left.\dfrac{d\Psi}{dt}\right|_{max}$

so, $\left.\dfrac{d\Psi}{dt}\right|_{max} = (\omega_0 \times 4\pi \times \pi \sin\omega_0 t)_{max}$

$V_{max} = (4\pi^2 \times 2\pi)$

$= 248.08$ volts

Thank you

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