# CALCULUS

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# Calculus

## Definition 1.1

*A function $'f'$ from a set $D$ to a set $Y$ is a rule that assigns a unique value $f(x)$ in $Y$ to each $x$ in $D$.*
*$D$ is called domain of $f$ and the set of all values of $f(x)$ is called range of $f(x)$. $Y$ is called codomain.*
*Clearly Range$(f(x)) \subseteq Y$.*

*i.e., if $y \in Y$, $y = f(x)$ is a function from $D$ to $Y$. $x$ is called independent variable, $y$ is called dependent variable. If both $x$ and $y$ are real numbers, then $f(x)$ is called a real valued function of a real variable.*

**Natural domain:** If no domain is given explicitly for a real valued function $f(x)$ of real variable, then the domain consists of all real numbers for which the $f(x)$ yields a real value. It is called natural domain.

## Example 1.2

*Find the domain and range of i)* $f(x) = 2 + \sqrt{x-1}$
*ii)* $f(x) = \frac{x+1}{x-1}$.

**Sol.** i) Since no domain is stated explicitly, domain of $f$ is natural domain i.e., $[1, \infty)$ [since for $x < 1$, $\sqrt{x-1}$ is not real].
Range of $2 + \sqrt{x-1}$ is $[2, \infty)$.
ii) Natural domain of $f(x) = \frac{x+1}{x-1}$ is $(-\infty, 1) \bigcup (1, \infty)$ [at $x = 1, f(x) \to \infty$].
If $y = \frac{x+1}{x-1} \Rightarrow x = \frac{y+1}{y-1}$.
$\therefore$ range is $(-\infty, 1) \bigcup (1, \infty)$.

# Families of curves

Constants that are allowed to vary are called **parameters**. And they make things more general in the geometrical sense.
Ex: $y = mx$ represents a straight line passing through origin if $m$ is fixed. But if it is allow to vary, then $y = mx$ produces infinite number of straight lines corresponding to all different values of $m$.

All these lines produced are called family of straight lines with the common property that is, "passing through origin". These parameters are also referred to as arbitrary constants.

Ex. $x^2 + y^2 = 4$ is the equation of a circle centred at $(0, 0)$. But $x^2 + y^2 = a^2$, where a is a parameter, is represents a family of all circles centred at $(0, 0)$ and different radii.

# Inverse functions

If the functions $f$ and $g$ satisfy the two conditions

$$g(f(x)) = x \text{ for every } x \text{ in domain}(f)$$
$$f(g(y)) = y \text{ for every } y \text{ in domain}(g),$$

then we say that $f$ is an inverse of $g$ and $g$ is an inverse of $f$.

## Example 1.3

$y = x^3 + 1, x = \sqrt[3]{y-1}$
*If* $f(x) = x^3 + 1$ *and* $g(y) = \sqrt[3]{y-1}$
$g(f(x)) = \sqrt[3]{f(x)-1} = \sqrt[3]{x^3+1-1} = x$
$f(g(y)) = (\sqrt[3]{y-1})^3 + 1 = y - 1 + 1 = y.$
$\therefore$ *$f$ is inverse if $g$ and vice-versa.*
$\therefore f^{-1}(f(x)) = x$ *and* $g^{-1}(g(y)) = y.$
*Clearly domain of* $(f^{-1})$ = *range of* $f$
*range* $(f^{-1})$ = *domain of* $(f)$.

# Method to find inverse of function

If the function $y = f(x)$, then solve for $x$ as a function $y$, say $x = g(y)$. Then $g(y)$ is inverse of $f$.

## Example 1.4

*Find inverse of* $f(x) = \sqrt{3x-2}$.
*Let* $y = \sqrt{3x-2} \Rightarrow x = \frac{1}{3}(y^2+2)$.
$\therefore f^{-1}(x) = \frac{1}{3}(x^2+2)$.

## Failure cases!

The above method of finding inverse fails in two reasons:
i) $f$ may not have inverse.
ii) $f$ may have inverse, but $x$ cannot be expressed explicitly as a function of $y$.
Thus finding conditions for existence of inverse of a function is important even if it cannot be found explicitly.

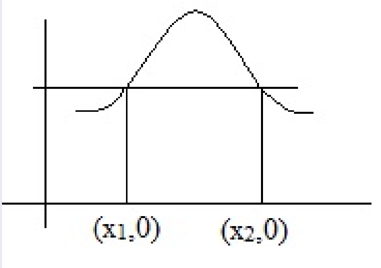
# Existence of inverse functions

## Theorem 1.5

*A function has an inverse if and only if it is one to one i.e., the function assigns distinct outputs to distinct inputs. [A function is one to one if and only if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$].*

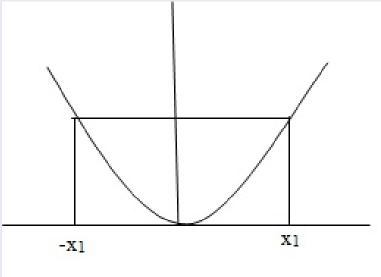
## Geometric Sense -Horizontal line test

$f$ is one to one if and only if the graph of $y = f(x)$ is cut atmost once by any horizontal line.



## Example 1.6

$f(x) = x^2$. *The graph is*

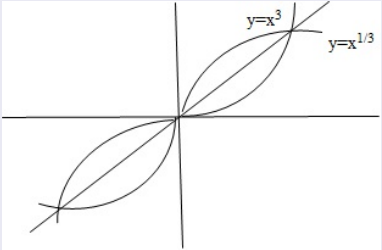


## Theorem 1.7

*If f has inverse, then the graphs of $y = f(x)$ an d $y = f^{-1}(x)$ are reflections of one another about $y = x$ i.e., one is the mirror image of the another with respect to $y = x$.*

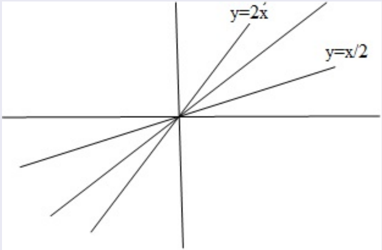
## Example 1.8

*Let* $y = x^3 \Rightarrow x = y^{1/3}$
*or* $f^{-1}(x) = g(x) = x^{1/3}$



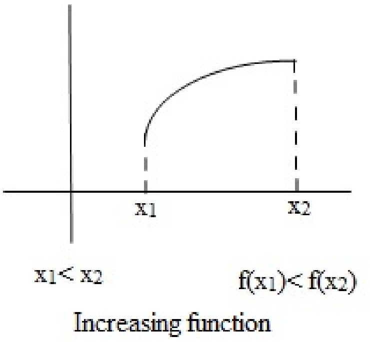
## Example 1.9

*Let* $y = 2x$ *or* $\frac{y}{2} = x$ *or* $f^{-1}(x) = y = \frac{x}{2}$



# Increasing and decreasing functions

If $x_1, x_2$ are points in the domain of $f$, then $f$ is increasing if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
Decreasing if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.



# Limits

Concept of "limit" is the fundamental building block and all concepts in calculus are based on this concept.
For basic understanding, consider the decimal expansion of $\frac{1}{3}$.

$\frac{1}{3} = 0.3333...$ which can be written as
$\frac{1}{3} = 0.3 + 0.03 + 0.003 + ...$ which is a sum of infinity many terms.
If $S_n =$ sum of the first 'n' terms,
$S_1 = 0.3 \quad S_2 = 0.33 \quad S_3 = 0.333... \quad S_n = 0.33...3.$

$S_n$ gets closer to a limit value of $\frac{1}{3}$ as $n$ increases.
**Note:** The most basic use of limits is to understand the behaviour of a function (dependent variable) as the independent variable approaches a given value.
Ex. $\lim_{x \to 3} (x^2 - 2x + 1) = 4.$

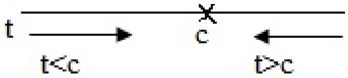
## Definition 1.10

*The limit of $f(t)$ as $t$ approaches $c$ is the number 'L' if given any $\epsilon > 0$, there exist $\delta > 0$ such that*

$$0 < |t - c| < \delta \quad |f(t) - L| < \epsilon.$$

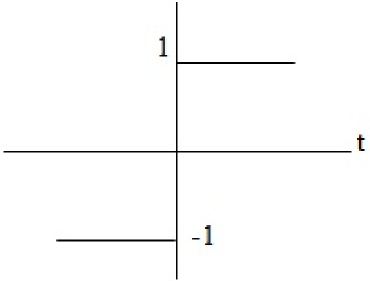
*It is written as $\lim_{t \to c} f(t) = L$.*
*Informally, if this value of $f(t)$ can be made as close as we like to $L$, by assuming $t$ to be sufficiently close to 'c',*
*then we can write $\lim_{t \to c} f(t) = L$.*

The above limit is called two-sided limit i.e., 't approaches c' [where $t$ and $c$ are real numbers] can happened in two ways:
i. $t \to c$, from right $(t > c)$
ii. $t \to c$ from left $(t < c)$.



Some functions exhibit different behaviours on the two sides of 'c'.
For example, consider

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$




Clearly, as $t$ approaches 0 from left , $f$ approaches $-1$ and $t$ approaches 0 from right , $f$ approaches 1. These facts are denoted by
$\lim_{x \to 0^-} f(x) = -1$ and $\lim_{x \to 0^+} f(x) = 1.$

## Definition 1.11

*The limit of* $f(t)$ *as* $t$ *approaches 'c' from the left equals* $L$ *if given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $\forall t, \quad c - \delta < t < c \Rightarrow |f(t) - L| < \epsilon$.
*Similarly, The limit of* $f(t)$ *as* $t$ *approaches 'c' from the right equals* $L$ *if given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that*
$\forall t, \quad c < t < c + \delta \Rightarrow |f(t) - L| < \epsilon$.

The limit of $f(t)$ at 'a' exists if and only if both left limit and right limit exist at 'a' and are equal i.e., $\lim_{t \to a} f(t) = L$ if and only if $\lim_{t \to a^+} f(t) = \lim_{t \to a^-} f(t) = L$.

## Example 1.12

*Find the* $\lim x \to 0 \frac{|x|}{x}$
$\lim x \to 0^- \frac{|x|}{x} = \frac{-x}{x} = -1$ *and* $\lim x \to 0^+ \frac{|x|}{x} = \frac{x}{x} = 1$.
$\therefore$ *limit doesn't exist.*

## Example 1.13

*Find* $\lim\limits_{x\to 3} \frac{4x^2-17x+15}{x^2-x-6}$.

**Sol.**

$$\begin{aligned}
\lim_{x\to 3} \frac{4x^2-17x+15}{x^2-x-6} &= \lim_{x\to 0} \frac{4(3-x)^2-17(3-x)+15}{(3-x)^2-(3-x)-6} \\
&= \lim_{x\to 0} \frac{36-24x+4x^2-51+17x+15}{9-6x+x^2-3+x-6} \\
&= \lim_{x\to 0} \frac{x(4x-7)}{x(x-5)} = \lim_{x\to 0} \frac{4x-7}{x-5} \\
&= \frac{7}{5}.
\end{aligned}$$

## Example 1.14

*Evaluate* $\lim_{x\to 2}[x]$.

**Sol.** $\lim_{x\to 2^{+0}}[x] = 2$ and $\lim_{x\to 2^{-0}}[x] = 1$
since $x \to 2^{+0} \Rightarrow 2 < x < 3 \Rightarrow [x] = 2$ and
$x \to 2^{-0} \Rightarrow 1 < x < 2 \Rightarrow [x] = 1$.
$\therefore \lim_{x\to 2}[x]$ does not exist.

## Example 1.15

*Prove that* $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

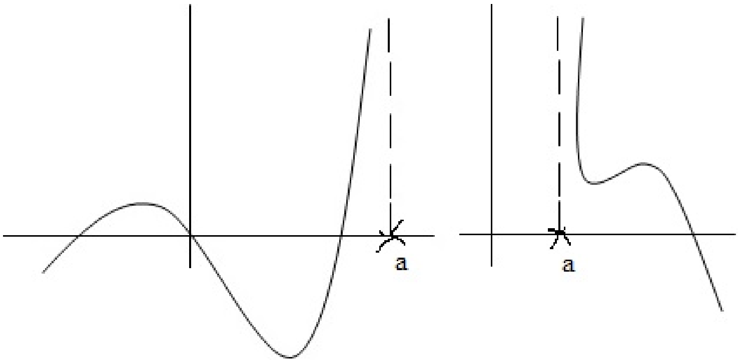
**Sol.** We cannot take the quotient of the limits since numerator oscillates between $-1$ and $1$, while the denominator increases without limit.
Since $x$ is +ve, $-1 \leq \sin x \leq 1 \Rightarrow \frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$
$\lim_{x\to\infty} \frac{-1}{x} \leq \lim_{x\to\infty} \frac{\sin x}{x} \leq \lim_{x\to\infty} \frac{1}{x}$. But $\lim_{x\to\infty} \frac{-1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x}$.
$\therefore \lim_{x\to\infty} \frac{\sin x}{x} = 0$. Similarly $\lim_{x\to\infty} \frac{\cos x}{x} = 0$.

# Exercise:

1. Evaluate $\lim\limits_{x \to 0} x \sin \frac{1}{x}$.
2. Evaluate $\lim\limits_{x \to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$.
3. Evaluate $\lim\limits_{x \to 2} x^x$.
4. Evaluate $\lim\limits_{x \to 0} \left(1 + \frac{2x}{3}\right)^{1/x}$.
5. Evaluate $\lim\limits_{x \to \infty} \frac{2^x+3^x}{3^x+1}$.

# Note:

If $\lim x \to af(x) = \infty$ or $-\infty$, then the line $x = a$ is called vertical asymptote.



# Important properties:

$\lim x \to af(x) = L_1$ and $\lim x \to ag(x) = L_2$, then

i. $\lim x \to a(f \pm g) = L_1 \pm L_2$.

ii. $\lim x \to a[f(x)g(x)] = L_1 L_2$.

iii. $\lim x \to a\left[\frac{f(x)}{g(x)}\right] = \frac{L_1}{L_2}, \quad L_2 \neq 0.$

iv. $\lim x \to a\sqrt[n]{f(x)} = \sqrt[n]{\lim x \to af(x)} = \sqrt[n]{L_1}, \quad L_1 > 0.$

v. $\lim x \to a\frac{x^n - a^n}{x - a} = na^{n-1}$(n is a non-zero rational numberand for all a for which $a^{n-1}$ is defined )

vi. $\lim x \to 0\frac{sinx}{x} = \lim x \to 0\frac{tanx}{x} = 1, 0 < |x| < \frac{\pi}{2}$

vii. $\lim x \to 0\frac{e^x - 1}{x} = 1$ and $\lim x \to 0\frac{a^x - 1}{x} = log_e a$

viii. $\lim x \to \infty[1 + \frac{1}{x}] = e = \lim x \to 0[1 + x]^{\frac{1}{x}}$

ix. $\lim_{x \to -\infty} \frac{1}{x} = 0$ and $\lim_{x \to \infty} \frac{1}{x} = 0$

x. $\lim_{x \to \pm\infty} (f(x))^n = \left[\lim_{x \to \pm\infty} f(x)\right]^n.$

xi. $\lim_{x \to \pm\infty} [kf(x)] = k \lim_{x \to \pm\infty} f(x).$

# Continuity

i) A function $y = f(x)$ is continuous at an interior point $'a'$ of a domain if

$$\lim_{x \to a} f(x) = f(a).$$

ii) If the domain is $[a, b]$, $y = f(x)$ is continuous at left end point $'a'$ if $\lim_{x \to a^+} f(x) = f(a)$. Similarly it is continuous at right end point $b$ if $\lim_{x \to b^-} f(x) = f(b)$.

iii) A $f(x)$ is said to be continuous if it is continuous at each point of its domain.

**Test for Continuity:** $y = f(x)$ is continuous at $x = a$ if and only if
i) $f(x)$ exists $\quad$ *ii*) $\lim_{x \to a} f(x)$ exists $\quad$ *iii*) $\lim_{x \to a} f(x) = f(a)$.

## Example 1.16

$f(x) = \frac{x^2-4}{x-2}$. *Discuss continuity at* $x = 2$.

$\lim\limits_{x\to 2} \frac{x^2-4}{x-2} = \lim\limits_{x\to 2}(x+2) = 4.$

## Example 1.17

$f(x) = |x|$. *Discuss continuity at* $x = 0$ *and* $x = a$.

$\lim\limits_{x\to 0} |x| = 0 = f(0).$

*at* $x = a,\ a > 0 \quad \lim\limits_{x\to a} |x| = \lim\limits_{x\to a} x = a = f(a).$

*For* $a < 0 \quad \lim\limits_{x\to a} |x| = \lim\limits_{x\to a} (-x) = -a = |a|.$

$\therefore$ $|x|$ *is continuous for all* $x$.

# Some properties:

1) If $f(x)$ and $g(x)$ are continuous functions at $x = a$, then
   i. $(f \pm g)$ is continuous at $x = a$.
   ii. $f.g$ is continuous at $x = a$.
   iii. $\frac{f}{g}$ is continuous at $x = a$ if $g(x) \neq 0$.
2) Polynomials are continuous everywhere.
3) A rational function is continuous at every point where the denominator is non-zero and has discontinuous at the points where denominator is zero.

**Theorem 1.18**

*If $f$ is one to one and is continuous at each point of its domain, then $f^{-1}$ is continuous at eqch point of its domain (range of $f$).*

## Example 1.19

*If* $f(x) = x\cos\frac{1}{x}, x \neq 0; f(x) = 0$ *for* $x = 0$*, examine the continuity at* $x = 0$*.*

**Sol.** Let $x \neq 0$. Then
$|f(x) - f(0)| = |f(x) - 0| = |x\cos\frac{1}{x}| = |x||\cos\frac{1}{x}| \leq |x|$
since $|\cos\frac{1}{x}| \leq 1$ for $x \neq 0$. $\therefore \quad 0 < |f(x) - f(0)| \leq |x|$.
But $\lim\limits_{x\to 0} 0 = 0$ and $\lim\limits_{x\to 0} |x| = 0$. $\therefore \quad \lim\limits_{x\to 0} f(x) = f(0)$.
$\therefore \quad f$ is continuous at $x = 0$.

## Example 1.20

*If* $f(x) = \begin{cases} \frac{e^{1/x}+1}{e^{1/x}-1} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$*, then examine the continuity of* $f$ *at '0'.*

**Sol.** $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left[\frac{e^{1/x}+1}{e^{1/x}-1}\right] = \lim\limits_{x\to 0}\left[\frac{1+e^{-1/x}}{1-e^{-1/x}}\right] = \frac{1+0}{1-0} = 1$
and $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}\left[\frac{e^{1/x}+1}{e^{1/x}-1}\right] = \frac{0+1}{0-1} = -1$.
$\therefore \lim\limits_{x\to 0} f(x)$ does not exists and hence $\lim\limits_{x\to 0} f(x) \neq f(0)$.
$\therefore \quad f(x)$ is discontinuous at '0'.

## Example 1.21

*Examine for continuity at* $x = 2$, $f(x) = \begin{cases} \frac{x^2}{2} - 2 & \text{for } 0 < x < 2 \\ 0 & \text{for } x = 2 \\ 2 - \frac{8}{x^2} & \text{for } x > 2 \end{cases}$

**Sol.** $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \left( \frac{x^2}{2} - 2 \right) = \frac{4}{2} - 2 = 0$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \left( 2 - \frac{8}{x^2} \right) = 2 - \frac{8}{4} = 0$

$\therefore \lim\limits_{x \to 2} f(x) = 0 = f(2) \Rightarrow f(x)$ is continuous at 2.

## Example 1.22

*If* $f(x) = x^{1/x-1}$ *is continuous at* $x = 1$, *find* $f(1)$.

**Sol.** $f(x)$ is continuous at $x = 1 \Rightarrow \lim\limits_{x \to 1} x^{1/x-1} = f(1)$.

$$\therefore f(1) = \lim_{x \to 1} e^{\log x^{1/x-1}} = \lim_{x \to 1} e^{\log x/(x-1)} = e^{\lim\limits_{x \to 1} \log x/(x-1)}$$

$$= e^{\lim\limits_{x \to 0} \log(1+x)/(1+x-1)} = e^{\lim\limits_{x \to 0} \frac{1}{x} \log(1+x)} = e^{\log e} = e.$$

# Exercise:

1. If $f(x) = \frac{x}{1+e^{1/x}}, x \neq 0$ and $f(0) = 0$, discuss the continuity at $x = 0$.
2. Find $k$, so that $f(x) = \frac{x^3+8}{x^5+32}$ for $x \neq -2$ and $f(x) = k$ for $x = -2$ is continuous at $x = -2$.
3. Examine the continuity of the function $f(x) = \frac{\cos 3x - \cos 4x}{x \sin 2x}, x \neq 0$; $f(x) = \frac{7}{4}$ when $x = 0$, at the point $x = 0$.
4. Let $f(x) = \frac{x^2}{2}, 0 < x < 1; f(x) = 2x^2 - 2x + \frac{3}{2}, 1 \leq x \leq 2$; discuss the continuity of $f$ on $[0, 2]$.
5. Show that the function $f(x) = |x| + |x-1| + |x-2|$ is continuous at $x = 0$.

# Differentiability

## Definition 1.23

*Let* $f : D \to \mathbb{R}, D \subseteq \mathbb{R}$ *and* $a \in D$. *Then* $f$ *is said to be differentiable at* $a$, *if*

$$L = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

*exists and* $L$ *is called derivative of* $f(t)$ *at* $x = a$ *and is denoted by* $f'(a)$.

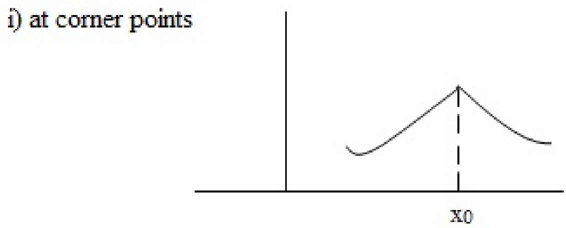
## Definition 1.24

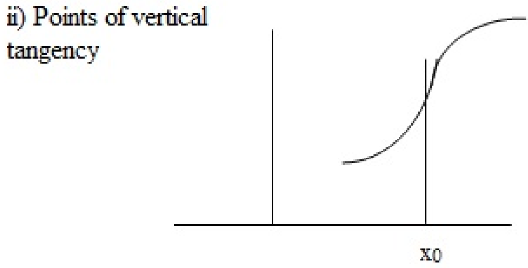
*Let* $f : D \to \mathbb{R}, D \subseteq \mathbb{R}$. $f$ *is said to be differentiable at* $x_0$ *if*

$$L = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, \quad h \in \mathbb{R}$$

*exists.*

A function which is continuous at $x_0$ may not be differentiable at $x_0$. Generally it happens in two ways



**Example 1.25**

*Discuss continuity and differentiability of* $f(x) = |x|$ *at* $x = 0$.

**Sol.** *If it exists, then*

$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = f'(0) = \lim_{h\to 0} \frac{|h|-0}{h} = \begin{cases} 1 & \text{if} \quad h > 0 \\ -1 & \text{if} \quad h < 0 \end{cases}$ $\therefore$ *the limits are not equal on left and right.*

$\therefore f$ *is not differentiable.*

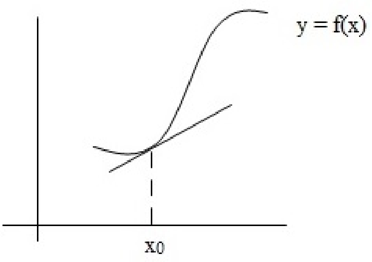
**Theorem 1.26**

*If a function is differentiable at* $x_0$*, then it is continuous at* $x_0$.

**Notation:** If the independent variable is $x$, then $f'(x) = \frac{df}{dx}$.

# Geometrical meaning

- If $y = f(x)$ is continuous function, then it can be represented as a smooth curve.
- $f'(x)$ at $x = x_0$ is the slope of the tangent line to the curve $f(x)$ at $x = x_0$.



- $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \qquad \text{if } x_0 + h = x \Rightarrow h = x - x_0$$
$$\Rightarrow (x - x_0) f'(x_0) \approx f(x) - f(x_0)$$
$$\text{or} \quad f(x) = y = f(x_0) + (x - x_0) f'(x_0) + \epsilon h$$
which is equation of tangent at $x_0$ and $f'(x) = m$,
then $y = f(x_0) + (x - x_0)m \quad (h \to 0)$.

## Example 1.27

*If* $f(x) = \frac{1-\cos x}{x}$ *for* $x \neq 0$; $\quad = 0$ *for* $x = 0$, *then find* $f'(0)$.

**Sol.** Now for $x \neq 0$, $f(x) - f(0) = \frac{1-\cos x}{x} - 0 = \frac{1-\cos x}{x} = \frac{2\sin^2 \frac{x}{2}}{x}$

$$\therefore \quad f'(0) = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{x^2} = \frac{1}{2} \left[ \lim_{x \to 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right]^2$$
$$= \frac{1}{2}(1)^2 = \frac{1}{2}.$$

## Example 1.28

*$f(x) = 3 + 2x$ for $-\frac{3}{2} < x \leq 0$ and $f(x) = 3 - 2x$ for $0 < x < \frac{3}{2}$. Show that $f(x)$ is continuous at $x = 0$ but $f'(0)$ does not exist.*

**Sol.** $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (3 - 2x) = 3$ and $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (3 + 2x) = 3.$

Also $f(0) = 3 + 2(0) = 3.$ $\quad \therefore \lim\limits_{x \to 0} f(x) = 3 = f(0)$

$\therefore f(x)$ is continuous at $x = 0.$

Also $f'(0) = \lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0}.$

Now $\quad \lim\limits_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^+} \frac{3 - 2x - 3}{x} = \lim\limits_{x \to 0^+} (-2) = -2$

and $\quad \lim\limits_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^-} \frac{3 + 2x - 3}{x} = \lim\limits_{x \to 0^-} (2) = 2.$

$\therefore \quad \lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist and hence $f'(0)$ does not exist.

**Exercise:**

1. If $f(x) = \frac{x - 1}{2x^2 - 7x + 5}$ for $x \neq 1; = -\frac{1}{3}$ for $x = 1$, then find $f'(1)$.
2. Is $f(x) = x|x|$ differentiable at $x = 0$.

# Increasing and decreasing nature of functions

**Theorem 1.29**

*Let $f(x)$ be function that is continuous on $[a, b]$ and differentiable on $(a, b)$.*

a) *If $f'(x) > 0 \forall x \in (a, b)$, then $f$ is increasing in$[a, b]$.*
b) *If $f'(x) < 0 \forall x \in (a, b)$, then $f$ is decreasing in$[a, b]$.*
c) *$f'(x) = 0 \forall x \in (a, b)$, then $f$ is constant in$[a, b]$.*

**Example 1.30**

*Find the interval on which $f(x) = x^2 - 4x + 3$ is increasing and the intervals in which it is decreasing.*

**Sol.** $f'(x) = 2x - 4$

$f'(x) < 0 \Rightarrow x < 2$ and $f'(x) > 0 \Rightarrow x > 2$.

$\therefore f$ is decreasing in $(-\infty, 2)$

$f$ is increasing in $(2, \infty)$.

## Example 1.31

*On what intervals is the function $f(x) = x^3 - 12x + 1$ increasing? On what intervals is it decreasing?*

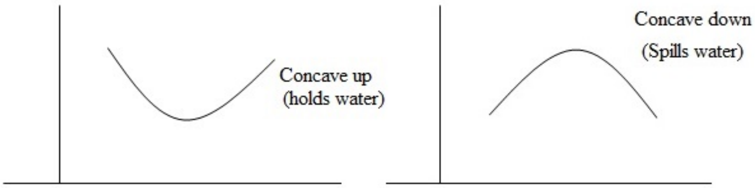
**Sol.** We have $f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$. Observe that $f'(x) > 0$ if $x < -2$ or $x > 2$ and $f'(x) < 0$ if $-2 < x < 2$. Therefore, $f$ is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$ and is decreasing on the interval $(-2, 2)$.

## Example 1.32

*Find the intervals of increase and decrease of the function $f(x) = x + \sin x$.*

**Sol.** We have $f'(x) = 1 + \cos x$. Observe that $f'(x) > 0$ for $x \in (-\infty, \infty)$. Therefore, $f$ is increasing on the interval $(-\infty, \infty)$.

# Concavity




## Definition 1.33

*If $f(x)$ is differentiable on $(a, b)$, then $f$ is said to be concave up on $(a, b)$ if $f'(x)$ is decreasing on $(a, b)$ and is concave down if $f'(x)$ is decreasing on $(a, b)$.*
*Concave up if $f''(x) > 0 \forall x \in (a, b)$*
*concave down if $f''(x) < 0 \forall x \in (a, b)$.*

## Example 1.34

*Discuss the concave nature of* $f(x) = x^2 - 4x + 3$ *in* $(-\infty, \infty)$.

**Sol.** $f'(x) = 2x - 4 \quad f''(x) = 2 > 0.$
$\therefore f$ is concave up.

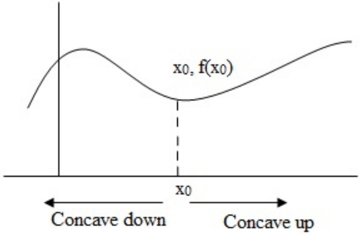
## Example 1.35

*Discuss the concave nature of* $f(x) = x^3 - 3x^2 + 1$ *in* $(-\infty, \infty)$.

**Sol.** $f'(x) = 3x^2 - 6x \quad f''(x) = 6x - 6.$
$f''(x) > 0 \Rightarrow x > 1$ Concave up in $(1, \infty)$
$f''(x) < 0 \Rightarrow x < 1$ Concave down in $(-\infty, 1)$.

# Inflection part:

The part$(x_0, f(x_0))$ at which the function changes the direction of its concavity is called inflection point of $f$.




In the above problem $f(x)$ is concave down in $(-\infty, 1)$ and concave up in $(1, \infty)$. $f$ changes its nature at $((x = 1, f(1))$ i.e., $(1, -1)$.

## Example 1.36

*Discuss the increasing, decreasing, concave up, down and locate inflection points of* $f(x) = x + 2sinx$ *in* $[0, 2\pi]$.

**Sol.** We have $f'(x) = 1 + 2\cos x$. Observe $f'(x) > 0$ i.e., $\cos x > \frac{-1}{2}$ if $\left(0, \frac{2\pi}{3}\right)$ or $\left(\frac{4\pi}{3}, 2\pi\right)$ and $f'(x) < 0$ if $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$. Therefore, $f$ is increasing on the intervals $\left(0, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$ and is decreasing on the interval $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$.

$f"(x) = -2\sin x$.

On the interval $(\pi, 2\pi), f"(x) > 0$ so $f$ is concave up. On the interval $(0, \pi), f"(x) < 0$ so $f$ is concave down. $f"(x)$ changes sign as we pass through $\pi$. $\therefore f$ has inflection points at $x = \pi$.

## Example 1.37

*Discuss the concave up, concave down and local inflection points of* $f(x) = x^3 - 12x + 1$.

**Sol.** $f'(x) = 3x^2 - 12$ and $f''(x) = 6x$. $f''(x) > 0$ in the interval $(0, \infty)$, so $f$ is concave up. $f''(x) < 0$ in the interval $(-\infty, 0)$, so $f$ is concave down. $\therefore f$ has inflection points at $x = 0$.

**Exercise:**

1. Discuss the increasing, decreasing, concave up, down and locate inflection points of $f(x) = x^4 - 2x^3 + 1$.
2. Discuss the increasing, decreasing, concave up, down and locate inflection points of $f(x) = \cos 3x$.
3. Discuss the increasing, decreasing, concave up, down and locate inflection points of $f(x) = xe^x$

**Definition 1.38**

*A function $F$ is called anti derivative of a function $f$ on a given $(a, b)$ if $F'(x) = f(x) \forall x \in (a, b)$.*

**Ex.** $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$.
$F'(x) = x^2$.

If $\frac{d}{dx}F(x) = f(x)$, then $\int f(x)dx = F(x) + c.$, $C$ is arbitrary constant.

# VII. INTEGRAL CALCULUS

## 1. Integration

(i) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1}\ (n \neq -1)$ $\qquad$ $\int \dfrac{1}{x}\,dx = \log_e x$

$\int e^x\,dx = e^x$ $\qquad$ $\int a^x\,dx = a^x/\log_e a$

(ii) $\int \sin x\,dx = -\cos x$ $\qquad$ $\int \cos x\,dx = \sin x$

$\int \tan x\,dx = -\log\cos x$ $\qquad$ $\int \cot x\,dx = \log\sin x$

$\int \sec x\,dx = \log(\sec x + \tan x)$ $\qquad$ $\int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x - \cot x)$

$\int \sec^2 x\,dx = \tan x$ $\qquad$ $\int \operatorname{cosec}^2 x\,dx = -\cot x$

(iii) $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$ $\qquad$ $\int \dfrac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\dfrac{x}{a}$

$\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\log\dfrac{a+x}{a-x}$ $\qquad$ $\int \dfrac{dx}{\sqrt{(a^2+x^2)}} = \sinh^{-1}\dfrac{x}{a}$

$\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\log\dfrac{x-a}{x+a}$ $\qquad$ $\int \dfrac{dx}{\sqrt{(x^2-a^2)}} = \cosh^{-1}\dfrac{x}{a}$

(iv) $\int \sqrt{(a^2-x^2)}\,dx = \dfrac{x\sqrt{(a^2-x^2)}}{2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}$

$\int \sqrt{(a^2+x^2)}\,dx = \dfrac{x\sqrt{(a^2+x^2)}}{2} + \dfrac{a^2}{2}\sinh^{-1}\dfrac{x}{a}$

$\int \sqrt{(x^2-a^2)}\,dx = \dfrac{x\sqrt{(x^2-a^2)}}{2} - \dfrac{a^2}{2}\cosh^{-1}\dfrac{x}{a}$

(v) $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$ $\qquad$ $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

(vi) $\int_0^\infty e^{-ax}\sin bx\,dx = \dfrac{b}{a^2+b^2}$ $\qquad$ $\int_0^\infty e^{-ax}\cos bx\,dx = \dfrac{a}{a^2+b^2}$

(vii) $\int \sinh x\,dx = \cosh x$ $\qquad$ $\int \cosh x\,dx = \sinh x$

$\int \tanh x\,dx = \log \cosh x$ $\qquad$ $\int \coth x\,dx = \log \sinh x$

$\int \operatorname{sech}^2 x\,dx = \tanh x$ $\qquad$ $\int \operatorname{cosech}^2 x\,dx = -\coth x$

(viii) $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \dfrac{(n-1)(n-3)(n-5)\ldots\ldots}{n(n-2)(n-4)\ldots\ldots} \times \left(\dfrac{\pi}{2}, \text{only if } n \text{ is even}\right)$

$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \dfrac{(m-1)(m-3)\ldots\ldots \times (n-1)(n-3)\ldots\ldots}{(m+n)(m+n-2)(m+n-4)\ldots\ldots} \times \left(\dfrac{\pi}{2}, \text{only if both } m \text{ and } n \text{ are even}\right)$

(ix) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx,$ $\qquad$ if $f(x)$ is an *even function*

$= 0,$ $\qquad$ if $f(x)$ is an *odd function.*

$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx,$ $\qquad$ if $f(2a-x) = f(x)$

$= 0,$ $\qquad$ if $f(2a-x) = -f(x)$.

(x) *Rule of Integration by parts :* $\int u(x)\,v(x)\,dx = uv_1 - \int u'\,v_1\,dx$

(xi) *Leibnitz General Rule of Integration by parts :* $\int u(x)\,v(x)dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots$

where dashes denote differentiation of $u$ and number of subscripts denotes number of times $v$ is integrated.

# Definite integrals:

**Definition 1.39**

*A function $f$ is said to be integrable on $[a, b]$ if the*

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

*exists and doesn't depend on the chose of partition or $x_k^*$ in the sub intervals.*

*In such case, the summation is denoted by $\int\limits_a^b f(x)dx$.*

## Properties:

i. $\int\limits_a^a f(x)dx = 0$

ii. $\int\limits_a^b f(x)dx = -\int\limits_b^a f(x)dx$

iii. $\int\limits_a^b cf(x)dx = c\int\limits_a^b f(x)dx$

iv. $\int\limits_a^b (f(x) \pm g(x))dx = \int\limits_a^b f(x)dx \pm \int\limits_a^b g(x)dx$

v. If $c \in (a, b)$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

## Definition 1.40

*A function $f$ is defined on an interval is said to be bounded on the interval. If there is a positive number $M$ such that*

$$|f(x)| \leq M \text{ or } \quad -M \leq f(x) \leq M.$$

## Theorem 1.41

*If $f$ is not bounded on $[a, b]$, then it is not integrable on $[a, b]$.*

# Improper Integrals

Till now we focussed on integrals and definite integrals with continuous integrands and finite intervals of integration. Now we extend these concepts to integrals with infinite intervals of integration and integrands that become infinity with in the interval. Such integrals are called **improper integrals.**

Improper integrals arise in three different ways.

i. integrals of the form $\int_a^{\infty} f(x)dx$, $\int_{-\infty}^{0} f(x)dx$, $\int_{-\infty}^{\infty} f(x)dx$. (Infinite intervals).
ii. integrands having discontinuities with in the interval of integration of the form $\int_a^a \frac{dx}{x^2}$, $\int_1^a \frac{dx}{x-1}$, $\int_a^{\pi} tanxdx$.
iii. integrals involving both the above

# Methods of evaluation

## Integrals of the form $\int_a^{\infty} f(x)dx$

The given integral is expressed as

$$\int_a^{\infty} f(x)dx =_{b\to\infty}^{lim} \int_a^{b} f(x)dx.$$

If the limit exists, then the improper integral is said to converge and the limit is the value of the integral. If the limit does not exist, then the improper integral is said to diverge and the integral is not assigned a value.

## Example 1: For what values of p, does the integral $\int_1^\infty \frac{dx}{x^p}$ converge?

**Solution**:

$$\begin{aligned}\int_1^\infty \frac{dx}{x^p} &= \lim_{b\to\infty} \int_1^b \frac{dx}{x^p} \\ &= \lim_{b\to\infty} [\frac{1}{(1-p)x^{p-1}}]_1^b \\ &= \lim_{b\to\infty} \frac{[b^{1-p}-1]}{1-p}.\end{aligned}$$

If $p > 1$, then $b^{1-p} \to 0$ as $b \to \infty$., then $\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}$.
If $p < 1$, then $b^{1-p} \to \infty$ as $b \to \infty$. Limit is not finite.

## Example 2 Evaluate $\int_0^\infty (1-x)e^{-x}dx$.

**Solution**:

$$\begin{aligned} \int (1-x)e^{-x}dx &= xe^{-x}+c \\ \int_0^\infty (1-x)e^{-x}dx &= \lim_{b\to\infty} \int_0^b (1-x)e^{-x}dx \\ &= \lim_{b\to\infty} [xe^{-x}+c]_0^b = \lim_{b\to\infty} [be^{-b}] \end{aligned} \tag{2.1}$$

This limit s of the form $\frac{\infty}{\infty}$. By L'Hospital's rule, this limit is 0. Therfore $\int_0^\infty (1-x)e^{-x}dx = 0$.

## Integrands having infinite discontinuities

If $f$ is continuous on $[a, b]$ except for an infinite discontinuity at $b$, then

$$\int_a^b f(x)dx = \lim_{k \to b^-} \int_a^k f(x)dx.$$

If the limit exists, then the improper integral is said to converge and the limit is the value of the integral.

## Example 1. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x}}$.

**Solution: Clearly, at $x = 1$, $\frac{dx}{\sqrt{1-x}} \to \infty$. Therefore**

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{k\to 1^-} \int_0^k \frac{dx}{\sqrt{1-x}}$$

$$= \lim_{k\to 1^-} [-2\sqrt{1-x}]_0^k = 2.$$

Example 2 Evaluate $\int_1^4 \frac{dx}{(x-2)^{\frac{2}{3}}}$

**Solution: Let I be the given integral. Therfore**

$$I = \int_1^2 \frac{dx}{(x-2)^{\frac{2}{3}}} + \int_2^4 \frac{dx}{(x-2)^{\frac{2}{3}}}$$

$$= \lim_{k\to 2^-} \int_1^k \frac{dx}{(x-2)^{\frac{2}{3}}} + 3[(x-2)^{\frac{1}{3}}]_2^4$$

$$I = 3(1+2^{\frac{1}{3}}).$$

## Do it!

1. Evaluate $\int_0^\infty e^{-x} dx$.
2. Evaluate $\int_{-\infty}^2 \frac{dx}{x^2+4}$.
3. Evaluate $\int_0^1 \frac{dx}{2x-1}$.
4. Find a positive value of 'a' such that $\int_0^\infty \frac{dx}{x^2+a^2} = 1$.

# Lengths, areas and volumes using integration

# Length of an arc

Let $y = f(x)$ be a smooth curve on $[a, b]$.$[f'$ be continuous on $[a, b]]$. Then the length of the curve over interval is called **arc length of $f$ over $[a, b]$.**

If $y = f(x)$ be a smooth curve on $[a, b]$, then **arc length L of $f$ over $[a, b]$ is**

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

**If the curve is in the form $x = g(y)$ over $[c, d]$, then**

$$L = \int_c^d \sqrt{1 + [g'(y)]^2} dy.$$

## In parametric form

If the curve is in the parametric form $x = x(t), y = y(t)$ with $t$ varying from $t_1$ to $t_2$, then

$$L = \int_{t_1}^{t_2} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt.$$

## In polar form

If the curve is in the polar form $r = f(\theta)$ with $\theta$ varying from $\theta_1$ to $\theta_2$, then

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta.$$

If the curve is in the polar form $\theta = f(r)$ with $r$ varying from $r_1$ to $r_2$, then $L = \int_{r_1}^{r_2} \sqrt{1 + r^2(\frac{d\theta}{dr})^2} dr.$

**Example 1 .** Find the length of the curve $8x = y^4 + 2y^{-2}$ from $y = 1$ to $y = 2$.

**Solution:Since the curve is in the form** $x = g(y)$, $\frac{dx}{dy} = \frac{1}{2}(y^3 - \frac{1}{y^3})$. **Therefore**

$$L = \int_1^2 \sqrt{1 + \frac{1}{4}(y^3 - \frac{1}{y^3})^2} = \frac{1}{2}[\frac{y^4}{4} - \frac{y^{-2}}{2}]_1^2 = \frac{33}{16}.$$

**Example 2 .** Find the length of the curve $x = e^{-t}cost, y = e^{-t}sint, 0 \leq t \leq \frac{\pi}{2}$.

**Solution: Clearly,** $\frac{dx}{dt} = -e^{-t}[cost + sint] and \frac{dy}{dt} = e^{-t}[cost - sint]$.

$(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 2e^{-2t}$.
And $L = \int_0^{\frac{\pi}{2}} \sqrt{2e^{-2t}}dt = \sqrt{2}(1 - e^{-\frac{\pi}{2}})$.

Example 3 . Find the length of the spiral $r = e^{a\theta}$ from the pole to the point $(r, \theta)$.

**Solution:** $\frac{dr}{d\theta} = ae^{a\theta}$ **and** $\sqrt{r^2 + (\frac{dr}{d\theta})^2} = e^{a\theta}\sqrt{1 + a^2}$. **And**

$$\begin{aligned} L &= \sqrt{1 + a^2} \int_0^{\theta} e^{a\theta} d\theta \\ &= \frac{\sqrt{1 + a^2}}{a}(e^{a\theta} - 1) \\ &= \frac{\sqrt{1 + a^2}}{a}(r - 1). \end{aligned}$$

## Rolle's Theorem

Consider $f : [a, b] \to R$. Suppose that (a) $f$ is continuous at every point of the closed interval $[a, b]$,
(b) $f$ is differentiable on $(a, b)$, and
(c) $f(a) = f(b)$. Then there is at least one number $c \in (a, b)$ such that $f'(c) = 0$.

## Example 1:

Verify Rolle's theorem and find an appropriate constant c for the function $\sqrt{1 - x^2}$ in $[-1, 1]$.

## Solution

Clearly $f$ is continuous on $[-1, 1]$ and $(-1, 1)$ respectively and $f(-1) = f(1)$.
Therefore, by Rolle's theorem there exists a $c \in [-1, 1]$ such that $f'(c) = 0$. $f'(c) = -\frac{c}{\sqrt{1-c^2}} = 0$.

## Example 2:

Verify Rolle's theorem for the function $f(x) = x^3 + x^2 - x + 1$ in the interval $[-1, 1]$.

## Solution

$f(x)$ is differentiable and therefore continuous for all $x$. Also

$$f(-1) = 2, f(1) = 2, \quad \therefore f(-) = f(1).$$

So that all the three conditions of the Rolle's theorem are satisfied in $[-1, 1]$. Also

$$f'(x) = 3x^2 + 2x - 1 = (3x - 1)(x + 1).$$

$\therefore\ f'(x) = 0$ for $x = \frac{1}{3}$ or $x = -1$, of which $\frac{1}{3}$ lies in $(-1, 1)$.
Thus there is a $c$, namely $c = \frac{1}{3}$ in $(-1, 1)$ for which $f'(c) = 0$.
This verifies Rolle's theorem.

## Exercise:

Verify Rolle's theorem for the following functions:

(i.) $f(x) = \frac{sin x}{e^x}$ in $[0, \pi]$

(ii.) $f(x) = (x - a)^m (x - b)^n$ in $[a, b], b > a, m > 1, n > 1.$

(iii.) $f(x) = x^2$ in $[-1, 1]$

(iv.) $f(x) = x(x + 3)e^{\frac{-x}{2}}$ in $[-3, 0]$.

## Lagrange's Mean Value Theorem:

Suppose that
(a) $y = f(x)$ is continuous on $[a, b]$,
(b) $f$ is differentiable on $(a, b)$. Then there is at least one number $c \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

## Example 1:

Verify the Lagrange's mean value theorem for function
$f(x) = lx^2 + mx + n,\ x \in [a, b]$.

## Solution

Since $f(x)$ is continuous and differentiable, there exists $c \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$. Therefore, $\frac{l(b^2-a^2)+m(b-a)}{b-a} = 2lc + m$.
Or $l(b + a) + m = 2lc + m$ which implies that $c = \frac{a+b}{2} \in [a, b]$.

## Example 2:

Verify Lagrange's mean value theorem for the function $f(x) = e^x$ in $[0, 1]$

## Solution:

Given function $f(x)$ is differentiable and therefore continuous for all $x$ and $f'(x) = e^x$.

Thus $f(x)$ satisfies both the conditions of Lagrange's mean value theorem.

$\therefore f'(c) = \frac{f(b)-f(a)}{b-a}$ holds for $e^c = \frac{e^1-e^0}{1} = e - 1$,

or $c = log(e - 1) = 0.5413 \in (0, 1)$.

## Example 3:

Verify Lagrange's mean value theorem for the function $f(x) = sin^{-1}x$ in $[0, 1]$.

## Solution:

$f(x)$ is differentiable in $(0, 1)$ and continuous for all x. Therefore both the conditions of the Lagrange's mean value theorem are satisfied in $[0, 1]$.
$\therefore \frac{1}{\sqrt{1-c^2}} = \frac{f(1)-f(0)}{1} = \frac{\pi}{2}$, or $c = \pm\frac{\sqrt{\pi^2-4}}{\pi} = 0.7712$ lies between 0 and 1.
Thus Lagrange's mean value theorem is verified.

## Example 4:

If $x > 0$ prove that $\frac{x}{1+x} < log(1+x) < x$.

## Solution:

From Lagrange's mean value theorem for the function $f(x) = log(1+x)$ in the interval $[0, x]$, we have $f(0) = 0, f'(x) = \frac{1}{1+x}$

$$\begin{aligned} f(x) &= f(0) + xf'(\theta x), \ \ for \ 0 < \theta < 1, \\ log(1+x) &= \frac{x}{1+\theta x} \end{aligned} \tag{3.1}$$

Now since $x > 0$ and $0 < \theta < 1$, we have

$$\begin{aligned} & x > \theta x > 0 \text{ so that} \\ & 1 + x > (1+\theta x) > 1 \text{ or} \\ & \frac{x}{1+x} < \frac{x}{1+\theta x} < x. \end{aligned} \tag{3.2}$$

Since $\frac{x}{1+\theta x} = log(1 + x)$ by expression (3.1), the inequality (3.2) becomes the result.

## Exercise:

Verify Lagrange's mean value theorem for the following functions.

(i.) $f(x) = logx$ in $[e, e^2]$.

(ii.) $f(x) = x(x - 1)(x - 2)$ in the interval $[0, \frac{1}{2}]$.

(iii.) $f(x) = cos^2 x$ in $[0, \frac{\pi}{2}]$.

(iv.) If $x > 0$ prove that $x - \frac{x^2}{2} < log(1 + x) < x - \frac{x^2}{2(1+x)}$

(v.) $f(x) = tan^{-1} x$ in $[0, 1]$.

## Taylor's Theorem (Generalized mean value theorem:)

Suppose a function $f(x)$ satisfies the following two conditions:

(i.) $f(x)$ and its first $n-1$ derivatives are continuous in [a,b].

(ii.) $f^{(n-1)}(x)$ is differentiable in (a,b).

Then there exist at least one point $c$ in (a,b) such that

$$\begin{aligned} f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \ldots \\ + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{(n)!}f^{(n)}(c) \end{aligned} \tag{3.3}$$

## Maclaurin's theorem:

Put b=x,a=0 in (3.3), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^n}{n!}f^(n)(\theta x).$$

Which is known as Maclaurin's expansion of $f(x)$ about the point $x = 0$.

## Example 1:

Prove that $tan^{-1}x = tan^{-1}(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})}{1+\frac{\pi^2}{16}} - ...$

## Solution:

We have to expand $f(x) = tan^{-1}x$ in a series of increasing powers of $(x - \frac{\pi}{4})$.
The Taylor's series expansion is

$$f(x) = f(\frac{\pi}{4}) + (x - \frac{\pi}{4})f'(\frac{\pi}{4}) + \frac{(x - \frac{\pi}{4})^2}{2!} f''(\frac{\pi}{4}) + ... \tag{3.4}$$

Now, we have

$$f(x) = tan^{-1}x, \ \Rightarrow f(\frac{\pi}{4}) = tan^{-1}(\frac{\pi}{4}).$$

$$f'(x) = \frac{1}{1+x^2}, \Rightarrow f'(\frac{\pi}{4}) = \frac{1}{1+\frac{\pi^2}{16}}.$$

$$f''(x) = \frac{-2x}{(1+x^2)^2}, \Rightarrow f''(\frac{\pi}{4}) = \frac{-2(\frac{\pi}{4})}{(1+\frac{\pi^2}{16})^2},$$

and so on. Using these in (3.4), we get

$$tan^{-1}x = tan^{-1}(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})}{1+\frac{\pi^2}{16}}$$
$$-\frac{\pi}{4}\frac{(x-\frac{\pi}{4})^2}{(1+\frac{\pi^2}{16})^2} + \ldots$$

## Example 2:

Obtain Taylor's series expansion of $f(x) = cosx$ about $x = \frac{\pi}{3}$. Hence find an approximate value of $cos61^0$.

## Solution:

The Taylor's series expansion of $f(x) = cosx$ in powers of $(x - \frac{\pi}{3})$ is

$$f(x) = f(\frac{\pi}{3}) + \sum_{n=1}^{\infty} \frac{(x - \frac{\pi}{3})^n}{n!} f^{(n)}(\frac{\pi}{3}) \tag{3.5}$$

We note that $f(\frac{\pi}{3}) = \frac{1}{2}$ and $f^{(n)}(x) = cos(x + \frac{n\pi}{2})$.
There fore, we get

$$
\begin{aligned}
cosx &= \frac{1}{2} - (x - \frac{\pi}{3})sin\frac{\pi}{3} - \frac{1}{2}(x - \frac{\pi}{3})^2 cos\frac{\pi}{3} \\
&+ \frac{1}{3!}(x - \frac{\pi}{3})^3 sin\frac{\pi}{3} \\
&= \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) - \frac{\sqrt{1}}{4}(x - \frac{\pi}{3})^2 + ...
\end{aligned} \tag{3.6}
$$

$\therefore cos61^0 \approx 0.4848.$

## Example 3:

Expand $logx$ in powers of $(x-1)$ and hence evaluate $log_e 1.1$ correct to 4 decimal places.

## Solution:

Given $f(x) = logx$.

$$\begin{aligned} \therefore f(1) &= 0, \\ f'(x) &= \frac{1}{x} \Rightarrow f'(1) = 1 \\ f''(x) &= -\frac{1}{x^2} \Rightarrow f''(1) = -1 \\ f'''(x) &= \frac{2}{x^3} \Rightarrow f'''(1) = 2 \end{aligned}$$

and so on. Substituting these values in Taylor's series, we get

$$logx = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - ...$$

Now putting x=1.1, we get $log 1.1 \approx 0.0953$.

## Example 4:

Expand $e^{sinx}$ by Maclaurin's series up to the term containing $x^4$.

## Solution:

Given $f(x) = e^{sinx}$.

$$\begin{aligned}
\therefore f(0) &= 1, \\
f'(x) &= f(x)cosx, f'(0) = 1, \\
f''(x) &= f'(x)cosx - f(x)sinx, f''(0) = 1, \\
f'''(x) &= f''(x)cosx - 2f'(x)sinx - f(x)cosx, f'''(0) = 0, \\
f^{iv}(0) &= -3
\end{aligned}$$

and so on. Substituting these values in Maclaurins's series we obtain

$$e^{sinx} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + ...$$

## Exercise:

(i.) Obtain Taylor's series expansion of $f(x) = logcosx$ about the point $x = \frac{\pi}{3}$.

(ii.) Using Taylor's theorem express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $(x - 1)$.

(iii.) Expand $sinx$ in powers of $(x - \frac{\pi}{2})$. Hence find the value of $sin91^o$ correct to 4 decimal places.

(iv.) Obtain Maclaurin's series expansion of $f(x) = \sqrt{1 + sin2x}$

(v.) Obtain Maclaurin's series expansion of $xcosecx$

## Local maxima and minima:

A function $f(x)$ is said to have a maximum value at $x = a$, if there exists a small number $h$, such that if $f(a) > f(x) \forall x$ in $(a - h, a + h)$.
A function $f(x)$ is said to have a minimum value at $x = a$, if there exists a small number $h$, such that if $f(a) < f(x) \forall x$ in $(a - h, a + h)$.
The maximum and minimum values of a function together are called extreme values and the points at which the function attains the extreme values are called the extreme points of the function.
Any point at which the concavity of the function changes from up to down or vice versa is called a point of inflexion. At a point of inflexion either $\frac{d^2y}{dx^2} = 0$ or $\frac{d^2y}{dx^2}$ does not exist.

## Procedure for finding maxima and minima:

(i.) put the given function $= f(x)$

(ii.) Find $f'(x)$ and equate it to 0. Solve this equation and let its roots be a,b,c,...

(iii.) Find $f''(x)$ and substitute in it by turns $x = a, b, c, ...$
If $f''(a)$ is $-ve$, $f(x)$ is maximum at $x = a$.
If $f''(a)$ is $+ve$, $f(x)$ is minimum at $x = a$.

(iv.) Sometimes $f''(x)$ may be difficult to find out or $f''(x)$ may be zero at $x = a$. In such cases,
(a.) If $f'(x)$ changes sign from $-ve$ to $+ve$ as $x$ passes through $a$, $f(x)$ is minimum at $x = a$.
(b.)If $f'(x)$ does not change sign while passing through $x = a$, $f(x)$ is neither maximum nor minimum at $x = a$.

## Example 1:

Find the maximum and minimum values of $3x^4 - 2x^3 - 6x^2 + 6x + 1$ in $(0, 2)$.

## Solution:

Let $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$
Then $f'(x) = 6(x^2 - 1)(2x - 1)$
$\therefore f'(x) = 0$ when $x = \pm 1, \frac{1}{2}$.
So in the interval $(0, 2)$ can have maximum or minimum at $x = \frac{1}{2}$ or 1.
Now $f''(x) = 12(3x^2 - x - 1$ so that $f''(\frac{1}{2}) = -9, f''(1) = 12$.
$\therefore f(x)$ has maximum at $x = \frac{1}{2}$ and a minimum at $x = 1$.
Thus the maximum value $= f(\frac{1}{2}) = 2.4375$ and the minimum value $= f(1) = 2$.

## Example 2:

Show that $sinx(1+cosx)$ having maximum value at $x=\frac{\pi}{3}$.

## Solution:

Let $f(x)=sinx(1+cosx)$
Then $f'(x)=(1+cosx)(2cosx-1)$
$\therefore f'(x)=0$ when $cosx=\frac{1}{2}$ or $-1$ i.e., when $x=\frac{\pi}{3}$ or $\pi$.
Now $f''(x)=-sinx(4cosx+1)$ so that $f''(\frac{\pi}{3})=\frac{-3\sqrt{2}}{2}$ and $f''(\pi)=0$.
Thus $f(x)$ has a maximum at $x=\frac{\pi}{3}$.
Since $f''(\pi)$ is 0, let us see whether $f'(x)$ changes sign or not.
When $x$ is slightly $<\pi$, $f'(x)$ is $-ve$, then when $x$ is slightly $>\pi$, $f'(x)$ is again $-ve$. i.e., $f'(x)$ does not change sign as $x$ passes through $\pi$. So $f(x)$ is neither maximum nor minimum at $x=\pi$.

## Example 3:

Analyze the function $f(x) = 12x^5 - 45x^4 + 40x^3 + 5$ and classify the stationary points as maxima, minima and points of inflexion.

## Solution:

$f'(x) = 60x^4 - 180x^3 + 120x^2 = 0 \Rightarrow x = 0, 1, 2.$
$f''(x) = 240x^3 - 540x^2 + 240x$
At $x = 0, f''(x) = 0$. So $x = 0$ is neither a point of maximum nor minimum but it is a point of inflexion.
At $x = 1. f''(x) = -60$. So $x = 1$ is a point of maxima with maximum value 12.
At $x = 2, f''(x) = 240$. So $x = 2$ is a point of minima with minimum value -11.

## Example 4:

Show that the right circular cylinder of given surface (including the ends) and maximum volume is such that its height is equal to the diameter of the base.

## Solution:

Let $r$ be the radius of the base and $h$, the height of the cylinder.
Thus given surface

$$S = 2\pi rh + 2\pi r^2 \tag{3.7}$$

and the volume

$$V = \pi r^2 h. \tag{3.8}$$

Hence $V$ is a function of two variables $r$ and $h$. To express $V$ in terms of one variable only (say $r$,) we get

$$V = \frac{1}{2}Sr - \pi r^3$$
$$\therefore \frac{dV}{dr} = \frac{1}{2}S - 3\pi r^2.$$

For $V$ to be maximum or minimum, we must have $\frac{dV}{dr} = 0$.
i.e.,$r = \sqrt{\frac{S}{6\pi}}$.
Also $\frac{d^2V}{dr^2} = -6\pi r$, which is negative for $r = \sqrt{\frac{S}{6\pi}}$.
Hence $V$ is maximum for $r = \sqrt{\frac{S}{6\pi}}$.
i.e., for $6\pi r^2 = S = 2\pi rh + 2\pi r^2$ i.e., for $h = 2r$, which proves the required result.

## Exercise:

(i.) Find maximum and minimum values of $f(x) = (x-2)^4$.

(ii.) Test the curve $y = x^4$ for points of inflexion.

(iii.) Find maximum and minimum values of $f(x) = (x^2 - 8x + 16)e^x$

(iv.) A rectangular sheet of metal of length 6 meters and width 2 meters is given. Four equal squares are removed from the corners. The sides of this sheet are now turned up to form an open rectangular box. Find approximately, the height of the box, such that the volume of the box is maximum.

v. Show that the points of inflexion of the curve $y^2 = (x-a)^2(x-b)$ lie on the straight line $3x + a = 4b$.