Multi variable Calculus

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Multiple integrals

Double integrals

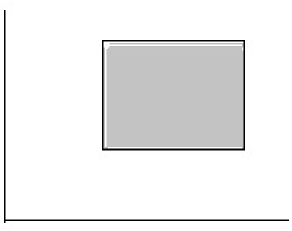
Consider a function $f(x, y)$ of two independent variables x and y, defined at each point in the finite region R of the xy plane.
The double integral of function $f(x, y)$ over a region R is defined as $\int\int_R f(x, y)dxdy$.

Domains

For single variable function $y = f(x)$, the domain is one dimensional.
For two variable function, $u = f(x, y)$, the natural domain is two dimensional region in a plane (xy-plane).
For three variable function, $u = f(x, y, z)$, the natural domain is three dimensional. i.e., in xyz-plane.

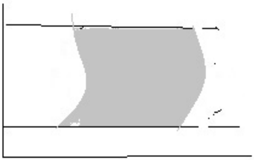
Different types of double integrals

1. Let $f(x,y)$ be a continuous function in R where $R = \{(x,y)/a \leq x \leq b; c \leq y \leq d\}$, then $\int\int_R f(x,y)dxdy = \int_{x=a}^{b}\left[\int_{y=c}^{d} f(x,y)dy\right]dx = \int_{y=c}^{d}\left[\int_{x=a}^{b} f(x,y)dx\right]dy.$



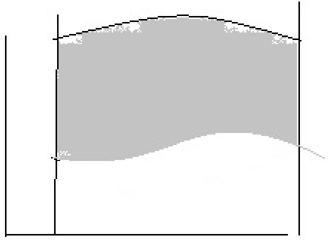
2. If $f(x, y)$ be a continuous function defined over the region R, where $R = \{(x, y)/a \leq x \leq b; y_1 \leq y \leq y_2\}$, then [here y_1 and y_2 are function of x and a, b are constants]

$$\int\int_R f(x, y)dxdy = \int_{x=a}^{b}\left[\int_{y=y_1}^{y_2} f(x, y)dy\right]dx.$$



3. If $f(x, y)$ be a continuous function defined over the region R, where $R = \{(x, y)/x_1 \leq x \leq x_2; c \leq y \leq d\}$, then [here x_1 and x_2 are function of y and c, d are constants.]

$$\int\int_R f(x, y)dxdy = \int_{y=c}^{d} \left[\int_{x=x_1}^{x_2} f(x, y)dx\right] dy.$$



Multiple Integrals

Note:1

If limits, for one variable is function of other and for other variable is constant, then first we have to integrate the variable with respect to variable for which limits are functions of the other, then the variable with constant limits.

That means limits of x are given as functions of y, then integrate w.r.t x first, then w.r.t y. Similarly limits of y are given as functions of x, then integrate w.r.t y first, then w.r.t x.

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$\int\int_R dA$ or $\int\int_R dxdy$ represents area of the region R.

Properties of double integrals:

i. $\int\int_R af(x,y)dxdy = a\int\int_R f(x,y)dxdy$

ii. $\int\int_R [f(x,y) \pm g(x,y)]dxdy = \int\int_R f(x,y)dxdy \pm \int\int_R g(x,y)dxdy$

iii. $\int\int_R f(x,y)dxdy = \int\int_{R_1} f(x,y)dxdy + \int\int_{R_2} f(x,y)dxdy$ where R is the union of R_1 and R_2.

Example 1.1

Evaluate $\int_0^1 \int_0^y e^{x/y} dxdy$.

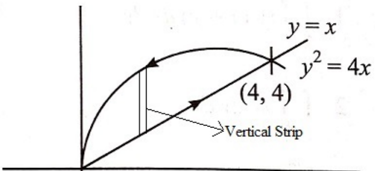
Sol.

$$\begin{aligned}
\int_0^1 \int_0^y e^{x/y} dxdy &= \int_0^1 \left[\int_0^y e^{x/y} dx \right] dy \\
&= \int_0^1 [ye^{x/y}]_{x=0}^{y} dy \\
&= \int_0^1 [ye - y] dy \\
&= (e-1) \left[\frac{y^2}{2} \right]_0^1 \\
&= \frac{1}{2}(e-1)
\end{aligned}$$

Example 1.2

Evaluate $\int\int xydxdy$ where R is the region bounded by $y = x, y^2 = 4x$.

Sol. The region R is bounded by the line $y = x$ and parabola $y^2 = 4x$.
The points of intersection of the line and parabola are given by $x^2 = 4x \Rightarrow x = 0, 4$.
If $x = 0, y = 0$ and if $x = 4, y = 4$.
$(0, 0)$ and $(4, 4)$ are points of intersection with the line $y = x$ and the parabola $y^2 = 4x$.



$$\therefore \int\int xydxdy = \int_0^4 \left[\int_x^{2\sqrt{x}} xydy\right] dx$$
$$= \int_0^4 \frac{x}{2}[y^2]_x^{2\sqrt{x}}$$
$$= \frac{1}{2}\int_0^4 x[4x - x^2]dx$$
$$= \frac{1}{2}\int_0^4 \left[\frac{4}{3}x^3 - \frac{x^4}{4}\right]_0^4$$
$$= \frac{32}{3}.$$

$$\int_{x=0}^{1}\left[\int_{y=x}^{\sqrt{x}}(x^2+y^2)\,dy\right]dx = \int_{x=0}^{1}\left[x^2y+\frac{y^3}{3}\right]_{y=x}^{\sqrt{x}}dx$$

$$= \int_{x=0}^{1}\left[\left(x^2.\sqrt{x}+\frac{(\sqrt{x})^3}{3}\right)-\left(x^2.x+\frac{x^3}{3}\right)\right]dx$$

$$= \int_{x=0}^{1}\left[x^{5/2}+\frac{x^{3/2}}{3}-\frac{4x^3}{3}\right]dx$$

$$= \left[\frac{x^{7/2}}{7/2}+\frac{x^{5/2}}{3.5/2}-\frac{4x^4}{4.3}\right]_0^1$$

$$= \frac{2}{7}+\frac{2}{15}-\frac{1}{3}=\frac{30+14-35}{105}=\frac{9}{105}=\frac{3}{35}$$

Example : Evaluate (*i*) $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy.$

(*ii*) $\int_0^4 \int_{y^2/4}^{y} \frac{y}{x^2+y^2}\,dx\,dy$

Solution : (*i*) $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy = \int_{x=0}^{5} \left[\int_{y=0}^{x^2} (x^3+xy^2)\,dy \right] dx$

$$= \int_0^5 \left(x^3 y + \frac{xy^3}{3} \right)_0^{x^2} dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3} \right) dx = \left(\frac{x^6}{6} + \frac{x^8}{24} \right)_0^5$$

$$= \frac{5^6}{6} + \frac{5^8}{24} = 5^6\left(\frac{1}{6}+\frac{25}{24}\right) = \frac{29(5^6)}{24}$$

(*ii*) Given integral $= \int_0^4 \left[y \int_{x=y^2/4}^{y} \frac{1}{x^2+y^2}\,dx \right] dy$ **[Here y is constant]**

$$= \int_0^4 \left[y \left\{ \frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) \right\}_{y^2/4}^{y} \right] dy \quad \left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \right]$$

$$= \int_0^4 \left[\tan^{-1}(1) - \tan^{-1}\left(\frac{y}{4}\right)\right] dy = \int_0^4 \left[\frac{\pi}{4} - \tan^{-1}\left(\frac{y}{4}\right)\right] dy$$

$$= \frac{\pi}{4}(y)_0^4 - \int_0^4 \tan^{-1}\frac{y}{4}\, dy$$ **[Integration by parts]**

$$= \pi - \left[\left\{\tan^{-1}\left(\frac{y}{4}\right).y\right\}_0^4 - \int_0^4 y \cdot \frac{1}{1+\frac{y^2}{16}} \cdot \frac{1}{4}\, dy\right]$$

$$= \pi - \left[4\tan^{-1}(1) - 4\int_0^4 \frac{y}{y^2+16}\, dy\right]$$

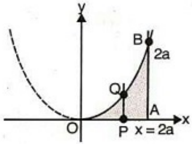
$$= \pi - \left[4\left(\frac{\pi}{4}\right) - 2\left\{\log(y^2+16)\right\}_0^4\right] \quad \left[\because \int \frac{f'(x)}{f(x)}\, dx = \log f(x)\right]$$

$$= 2[\log(32) - \log 16] = 2\log(32/16) = 2\log 2$$

Example : Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by x- axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.

Solution : Let us draw the parabola $x^2 = 4ay$, the line $x = 2a$ and identify the region R of integration. It is as in figure. The integral $\iint_R xy\,dx\,dy$ is same as $\iint_R xy\,dy\,dx$.

Let us consider a fixed x (Draw a line $x = k$ in the region). Now for this fixed x, y varies from 0 to $x^2/4a$. To be in the region, we have to vary x from 0 to $2a$.



Hence the given integral $= \int_{x=0}^{2a} \int_{y=0}^{x^2/4a} xy\,dy\,dx$

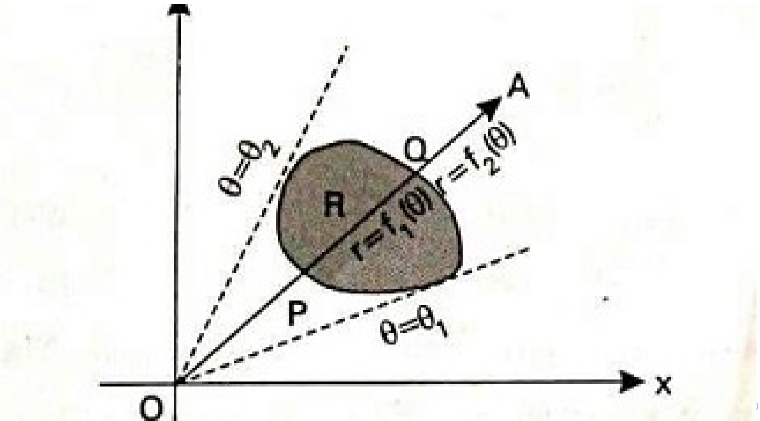
$$= \int_{x=0}^{2a} \left[\int_{y=0}^{x^2/4a} y\,dy \right] x\,dx = \int_{x=0}^{2a} \left[\frac{y^2}{2} \right]_{y=0}^{x^2/4a} x\,dx$$

$$= \int_{x=0}^{2a} \frac{x^4}{32a^2} x\,dx = \frac{1}{32a^2} \int_{x=0}^{2a} x^5\,dx$$

$$= \frac{1}{32a^2} \left(\frac{x^6}{6} \right)_{x=0}^{2a} = \frac{64\,a^6}{32.\,a^2.6} = \frac{a^4}{3}$$

Double integrals in polar form

Let the double integral is of the form $\int_{\alpha}^{\beta}\int_{r_1}^{r_2} f(r,\theta)drd\theta$.
Here $r_1 = f(\theta_1)$ and $r_2 = f(\theta_2), \theta = \alpha$ and $\theta = \beta$[α, β are constants]
then $\int_{\alpha}^{\beta}\int_{r_1}^{r_2} f(r,\theta)drd\theta = \int_{\alpha}^{\beta}\left[\int_{r=f(\theta_1)}^{r=f(\theta_2)} f(r,\theta)dr\right]d\theta$.



Procedure:

1. Sketch the region given by identifying the curves
2. Imagine the radius vector OA in the region and mark the limits of r as functions of θ say $r = f_1(\theta)$ to $f_2(\theta)$.
3. Find the smallest and largest values of θ between which the complete region lies.
4. Then the integral is evaluated as $\int_{\theta_1}^{\theta_2} \int_{f_1(\theta)}^{f_2(\theta)} f(r, \theta) dr d(\theta)$.

Ex.Evaluate $\int_0^{\pi/2}\int_0^{\infty}\frac{rdrd\theta}{(r^2+a^2)^2}$.

Sol. $\int_0^{\pi/2}\int_0^{\infty}\frac{rdrd\theta}{(r^2+a^2)^2} = \int_0^{\pi/2}\left[\int_0^{\infty}\frac{r}{(r^2+a^2)^2}dr\right]d\theta$

Put $r^2+a^2=6$

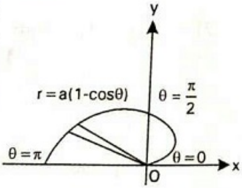
$2rdr = dt \to rdr = \frac{dt}{2}$. So

$$\int_0^{\pi/2}\int_0^{\infty}\frac{rdrd\theta}{(r^2+a^2)^2} = \int_0^{\pi/2}\left[\int_{a^2}^{\infty}\frac{dt}{2t^2}\right]d\theta = \int_0^{\pi/2}\left[\frac{-1}{2t}\right]_{a^2}^{\infty}$$

$$= \int_0^{\pi/2}\left[\frac{-1}{2a^2}\right]d\theta = \frac{-1}{2a^2}[\theta]_0^{\pi/2} = \frac{\pi}{4a^2}.$$

Example : Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the cardioid $r = a(1-\cos\theta)$ above the initial line.

Solution : The cardioid $r = a(1-\cos\theta)$ is symmetrical about the initial line and it passes through the pole O when $\theta = 0$. The region of integration R above the initial line is covered by radial strips whose ends are $r = 0$ and $r = a(1-\cos\theta)$, the strips starting from $\theta = 0$ and ending at $\theta = \pi$.



$$\therefore \iint_R r\sin\theta\, dr\, d\theta = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1-\cos\theta)} r\sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi}\sin\theta\left\{\int_0^{a(1-\cos\theta)} r\, dr\right\}d\theta = \int_0^{\pi}\sin\theta\left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} d\theta$$

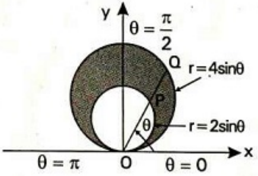
$$= \frac{1}{2}\int_0^{\pi}\sin\theta\, a^2(1-\cos\theta)^2 d\theta$$

$$= \frac{a^2}{2}\left[\frac{(1-\cos\theta)^3}{3}\right]_0^{\pi} = \frac{a^2}{6}\left[(1-\cos\pi)^3-(1-\cos 0)\right]$$

$$= \frac{a^2}{6}[8-0] = \frac{4a^2}{3}.$$

Example : Evaluate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Solution : The region of integration R is shown shaded. Here r varies from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π.



$$\therefore \iint r^3\, dr\, d\theta = \int_{\theta=0}^{\pi} \int_{r=2\sin\theta}^{4\sin\theta} r^3\, dr\, d\theta$$

$$= \int_0^{\pi} \left\{ \int_{r=2\sin\theta}^{4\sin\theta} r^3\, dr \right\} d\theta$$

$$= \int_0^{\pi} \left(\frac{r^4}{4} \right)_{2\sin\theta}^{4\sin\theta} d\theta$$

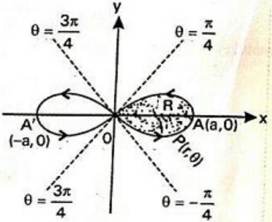
$$= \frac{1}{4} \int_0^{\pi} (256\sin^4\theta - 16\sin^4\theta)\, d\theta$$

$$= 60 \int_0^{\pi} \sin^4\theta\, d\theta \quad \left[\because \int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx, \text{if } f(2a-x) = f(x) \right]$$

Example : Evaluate $\iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$.

Solution :

The curve $r^2 = a^2 \cos 2\theta$ is symmetrical about the initial line (i.e. x - axis) and about the line $\theta = \frac{\pi}{2}$ (i.e. y-axis). The curve intersects the x-axis at the points $A(a, 0)$ and $A'(-a,0)$. Due to symmetry one loop is formed between the points O and A and the other between O and A'.



Imagine a radius vector from the pole O(r = 0) through the region R, which emerges at P where $r = a\sqrt{\cos 2\theta}$. Such radii can be drawn in between the lines $\theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$.

Hence $\iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}} = \int_{\theta=-\pi.4}^{\pi/4} \int_{r=0}^{a\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}} = \int_{\theta=-\pi/4}^{\pi/4} d\theta \cdot \frac{1}{2} \int_{r=0}^{a\sqrt{\cos 2\theta}} \frac{2r}{\sqrt{a^2+r^2}}\,dr$

$$= \frac{1}{2}\int_{\theta=-\pi/4}^{\pi/4} \left[\frac{\sqrt{a^2+r^2}}{1/2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta = \int_{\theta=-\pi/4}^{\pi/4} \left[\sqrt{a^2+a^2\cos 2\theta} - \sqrt{a^2}\right] d\theta$$

$$= a\int_{-\pi/4}^{\pi/4} \left[\sqrt{1+\cos 2\theta} - 1\right] d\theta = a\int_{-\pi/4}^{\pi/4} \left[\sqrt{2}\cos\theta - 1\right] d\theta = a\left[\sqrt{2}\sin\theta - \theta\right]_{-\pi/4}^{\pi/4}$$

Exercise:

1. Evaluate $\int_0^2 \int_0^{x^2} e^{y/x} dydx$.
2. Evaluate $\int\int_R ydxdy$ where R is bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.
3. Evaluate $\int_a^{\pi/2} \int_{a(1+\cos\theta)}^a rdrd\theta$.

Change the order of integration:

If, in the given double integral the integration is w.r.t. x and then w.r.t y, the process of converting the order of integration is called change of order of integration. Change of order of integration changes the limits of integration.

Consider a double integral $\int\int_R f(x, y)dxdy$ where R is region.
Assume that R lies between the lines $x = x_0, x = x_1$ and curves $y = f_1(x)$ and $y = f_2(x)$. For points of R, x lies in the interval $[x_0, x_1]$, y varies between $f_1(x)$ and $f_2(x)$, where $f_1(x)$ and $f_2(x)$ are the ordinates of the points at which the boundary of R is intersected by line through (x, y) and parallel to y-axis.

$$\int\int_R f(x,y)dxdy = \int_{x=x_0}^{x_1}\left[\int_{y=f_1(x)}^{f_2(x)} f(x,y)dy\right]dx$$

By change of order of integration, limits of y will be constants y_0, y_1 and x varies between $g_1(y)$ and $g_2(y)$ which the boundary is intersected by the line through (x, y) and parallel to x-axis.

$$\int\int_R f(x,y)dxdy = \int_{y=y_0}^{y_1}\left[\int_{x=g_1(y)}^{g_2(y)} f(x,y)dx\right]dy.$$

Change of order of integration:

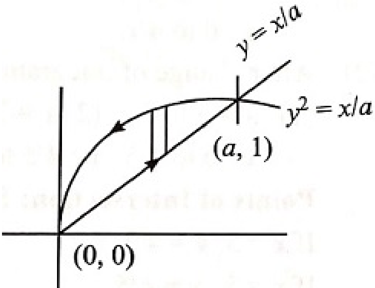
Procedure to change the order of integration:

1. Identify the variables for the limits.
2. Trace the curve.
3. If we are evaluating with respect to y first, then take strip parallel to y-axis.
   If the evaluation is with respect to x first, then take strip parallel to x-axis.
4. Rotate the strip to 90^0 in anti clock wise direction and identify the starting and ending points of the strip, which will be below and upper units of that variable.
5. Identify the limits for other variables for the region of consideration.
6. Evaluate the double integral with new order of integration.

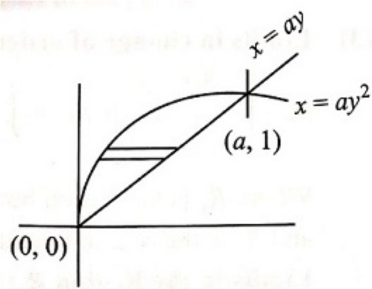
Example 1.3

Evaluate $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2)dydx$ by change of order of integration.

Sol. Before change of order in integration: $x : 0$ to $a, y : \frac{x}{a}$ to $\sqrt{\frac{x}{a}}$.
i.e., $x = 0, x = a$ and $x = ay, x = ay^2$.



After change of order of integration: $x : ay^2$ to ay, $y : 0$ to 1.



$$\therefore \int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2)\,dydx = \int_0^1 \left[\int_{ay^2}^{ay} (x^2 + y^2)\,dx\right] dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_{ay^2}^{ay} dy$$

$$= \int_0^1 \left[\frac{a^3}{3} y^3 + ay^3 - a^3 3y^6 - ay^4\right.$$

$$= \left[\frac{a^3}{3}\frac{y^4}{4} + \frac{a}{4}y^4 - \frac{a^3}{3}\frac{y^7}{7} - \frac{a}{5}y^5\right]_0^1$$

$$= \frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5}$$

$$= \frac{a^3}{28} + \frac{a}{20}.$$

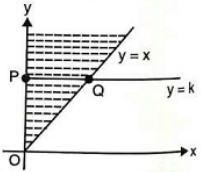
Example : Evaluate the integral by changing the order of integration $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

Solution : The given integral is $\int_{x=0}^{\infty}\left[\int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\right] dx$

Note that we cannot evaluate the integral within the square brackets. Let us change the order of integration and see what happens.

As is given, the region is described by fixing x first, finding the limits of y and then changing x.

For a fixed x, y is changing from x to ∞.



Draw the line $y = x$. Then x varies from 0 to ∞.

The region of integration is shaded in the figure.

Now let us change the order of integration. Suppose we fix y first. This, we shall do, by drawing line $y=k$.

Now x varies from 0 to y. To cover the region, then y has to vary from 0 to ∞. The integral now can be written as $\int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y} dx.\, dy$

This is equal to

$$\int_{y=0}^{\infty} \left[\int_{x=0}^{y} \frac{e^{-y}}{y} dx \right] dy = \int_{y=0}^{\infty} \left[\frac{e^{-y}}{y}.x \right]_{x=0}^{y} dy = \int_{y=0}^{\infty} \frac{e^{-y}}{y}.y.\, dy$$

$$= \int_{y=0}^{\infty} e^{-y} dy = \left[\frac{e^{-y}}{-1} \right]_{y=0}^{\infty} = 0-(-1) = 1$$

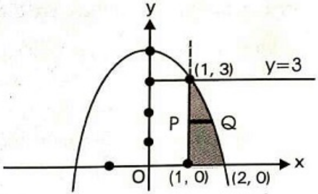
(Notice that while we were unable to evaluate the double integral as was given, we could evaluate it, easily, by changing the order of integration).

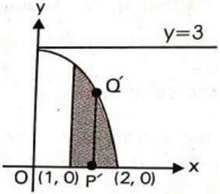
Example : By Changing the order of integration, evaluate $\int_0^3 \int_1^{\sqrt{(4-y)}} (x+y)\, dx\, dy$.

Solution : The given integral is $\int_{y=0}^{3} \int_{x=1}^{\sqrt{(4-y)}} (x+y)\, dx\, dy$

For a fixed y, x varies from 1 to $\sqrt{4-y}$ and then y varies from 0 to 3. Let us draw the curves $x=1$ and $x=\sqrt{(4-y)}$ (*i.e.*) $x=1$ and $x^2=4-y$. $x=1$ is a line parallel to y axis. $x^2=4-y$ is a curve symmetric about y-axis and (0, 4) is a point on it. It meets x axis at $(\pm 2, 0)$. For $y>4$ there are no points on the curve. $x=1$ and $x^2=4-y$ intersect at (1, 3). In view of the above description, we identify that the region of integration is the shaded region.

Let us fix x. For a fixed x, y varies from 0 to $4-x^2$. Then x varies from 1 to 2 to get the region of integration. The integral is $\int_{x=1}^{2} \int_{y=0}^{4-x^2} (x+y)\, dy\, dx$





This is equal to

$$\int_{x=1}^{2}\left[xy+\frac{y^2}{2}\right]_{y=0}^{4-x^2} dx = \int_{x=1}^{2}\left[x(4-x^2)+\frac{(4-x^2)^2}{2}\right] dx$$

$$=\left[4\frac{x^2}{2}-\frac{x^4}{4}+\frac{1}{2}\left(16x+\frac{x^5}{5}-\frac{8x^3}{3}\right)\right]_1^2$$

$$=\left[8-4+\frac{1}{2}(32+\frac{32}{5}-\frac{64}{3})\right]-\left[2-\frac{1}{4}+\frac{1}{2}(16+\frac{1}{5}-\frac{8}{3})\right]$$

$$=\frac{241}{60}.$$

Example : Evaluate by changing the order of integration $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dydx}{\sqrt{y^4 - a^2x^2}}$.

Solution : Here y varies from $\sqrt{ax}$ to a and x varies from 0 to a.

i.e., Limits are $y = \sqrt{ax}$ or $y^2 = ax, y = a$ and $x = 0, x = a$.

Hence it is clear that the integration is performed first w.r.t. 'y' and then w.r.t. 'x'.

Thus problem is first performed along the vertical strip PQ which starts from $y^2 = ax$ to $y = a$.

Hence the region of integration is the dotted portion OABO.

MULTI-VARIABLE CALCULUS

Prof. GVSR Deekshitulu

On changing the order of integration, we first integrate w.r.t. 'x' along a horizontal strip RS which extends from $x=0$ to $x=y^2/a$.

To cover the region, we then integrate w.r.t. 'y' from $y=0$ to $y=a$.

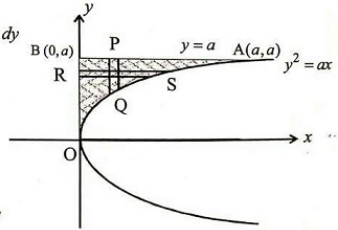
$$\therefore \int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dy\, dx}{\sqrt{y^4-a^2x^2}} = \int_0^a \int_0^{y^2/a} y^2 \cdot \frac{1}{\sqrt{y^4-a^2x^2}}\, dx\, dy$$

$$= \frac{1}{a}\int_0^a y^2 \int_0^{y^2/a} \frac{1}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}}\, dx\, dy$$

$$= \frac{1}{a}\int_0^a y^2 \left(\sin^{-1}\frac{ax}{y^2}\right)_0^{y^2/a} dy$$

$$= \frac{1}{a}\int_0^a y^2 (\sin^{-1}1 - \sin^{-1}0)\, dy$$

$$= \frac{\pi}{2a}\int_0^a y^2 dy = \frac{\pi}{2a}\left(\frac{y^3}{3}\right)_0^a = \frac{\pi}{2a}\left(\frac{a^3}{3}-0\right) = \frac{\pi a^2}{6}$$



Exercise:

1. By change of order of integration, evaluate $\int_0^a \int_x^a (x^2 + y^2) dydx$.
2. By change of order of integration, evaluate $\int_0^a \int_{x^2/a}^{2a-x} xy^2 dydx$.

Change of variables in double integrals:

Change of variables from cartesian (x, y) to polar form (r, θ):

Consider the double integral in the region R, $\int\int_R f(x, y)dxdy$ in the xy plane.
Put $x = r\cos\theta$ and $y = r\sin\theta$, then
$\int\int_R f(x, y)dxdy = \int\int_{R*} f(r\cos\theta, r\sin\theta)|J|drd\theta$
$= \int\int_{R*} f(r\cos\theta, r\sin\theta)rdrd\theta$,

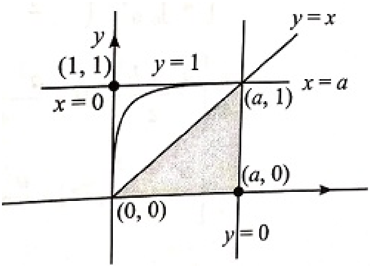
where J is the Jacobian and

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Example 1.4

Evaluate $\int_0^1 \int_y^a \frac{x}{x^2+y^2}dxdy$ by changing into polar coordinates.

Sol.



The region of integration bounded by $y = 0$ to $y = 1$ and $x = y$ to $x = a$.
From the figure, it is located in first quadrant only.

Changing into polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rdrd\theta$.

The limits for r and θ are obtained by

$x = y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \tan\theta = 1$ or $\theta = \frac{\pi}{4}$.

$x = a \Rightarrow r\cos\theta = a \Rightarrow r = \frac{a}{\cos\theta}$.

$\therefore r$ varies from 0 to $\frac{a}{\cos\theta}$, θ varies from 0 to $\frac{\pi}{4}$

$$\begin{aligned}\therefore \int_0^1\int_y^a \frac{x}{x^2+y^2}dxdy &= \int_0^{\pi/4}\int_0^{a/\cos\theta}\frac{r\cos\theta}{r^2}rdrd\theta \\ &= \int_0^{\pi/4}\int_0^{a/\cos\theta} cos\theta drd\theta \\ &= \int_0^{\pi/4}[r]_0^{a/\cos\theta}\cos\theta d\theta \\ &= \int_0^{\pi/4} ad\theta = \frac{a\pi}{4}\end{aligned}$$

Example : Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ by changing into polar coordinates.

Solution : The region of integration is $y=0, y=\sqrt{2x-x^2}, x=0, x=2$

i.e., $y=0, x^2+y^2=2x, x=0, x=2$ ... (1)

We know that $x^2+y^2=2x$ represents the circle with centre (1,0) and radius 1.

In polar co-ordinates the same region is bounded by the curves $r=0, r=2\cos\theta, \theta=0$ and $\theta=\pi/2$

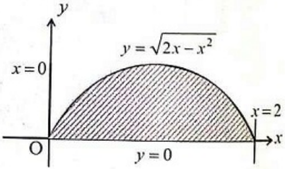
[By putting $x=r\cos\theta, y=r\sin\theta$ in (1), we get

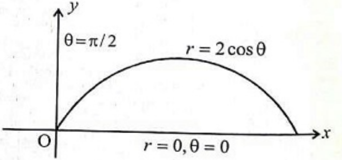
$y=0 \Rightarrow r\sin\theta=0 \Rightarrow \theta=0 \quad (\because r\neq 0)$

$x=0 \Rightarrow r\cos\theta=0 \Rightarrow \theta=\pi/2 \quad (\because r\neq 0)$

$y=\sqrt{2x-x^2} \Rightarrow x^2+y^2=2x \Rightarrow r^2=2r\cos\theta$

$\Rightarrow r=2\cos\theta$]





$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} \frac{(r\cos\theta)}{r} \,.\, r\,dr\,d\theta \quad [\because\ dx\,dy = r\,dr\,d\theta]$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} r\cos\theta\,dr\,d\theta = \int_0^{\pi/2} \cos\theta \left(\frac{r^2}{2}\right)_0^{2\cos\theta} d\theta$$

$$= 2\int_0^{\pi/2} \cos^3\theta\,d\theta = 2.\frac{2}{3} = \frac{4}{3}$$

Exercise

1. Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)dxdy$ by changing into polar coordinates.
2. Evaluate $\int_0^\infty \int_0^\infty \frac{dxdy}{(x^2+y^2+a^2)^2}$ using polar coordinates.

Triple integrals:

Consider a region v in three dimensional xyz-space. Let $f(x, y, z)$ be a continuous function of three variables over the region v .

1. Let $f(x, y, z)$ be a continuous function over a regular solid v defined by $a < x < b, c < y < d, e < z < f$. Then $\underset{V}{\int\int\int} f(x, y, z)dxdydz = \int_a^b \int_c^d \int_e^f f(x, y, z)dzdydx$.
2. If $f(x, y, z)$ be continuous function defined by $a < x < b, f_1(x) < y < f_2(x), g_1(x, y) < z < g_2(x, y)$ then $\underset{V}{\int\int\int} f(x, y, z)dxdydz =$ $\int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)dzdydx$.

Example 1.5

Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$.

Sol. We have $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a)$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{(\sqrt{1-x^2-y^2})^2 - z^2}} dydx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \sin^{-1}\left[\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dydx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dydx$$

$$= \int_0^1 \frac{\pi}{2}[y]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\frac{x}{1}\right]_0^1$$

$$= \frac{\pi}{2}\left[\frac{1}{2}.\frac{\pi}{2}\right]$$

$$= \frac{\pi^2}{8}.$$

Example 1.6

Evaluate $\int\int\limits_{V}\int xyzdxdydz$ where v is bounded by the coordinate planes and the plane $x + y + z = 1$.

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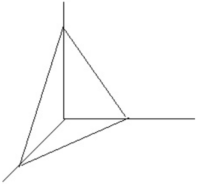


Sol. The region is bounded by the planes $x = 0, y = 0, z = 0$ and the plane $x + y + z = 1$.

i.e., z varies from 0 to $1-x-y$

y varies from 0 to $1-x$ [in xy-plane $z=0$ i.e., $x+y=1$]

x varies from 0 to 1.

$$\iiint_V xyz\,dx\,dy\,dz = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} xy\left[\frac{z^2}{2}\right]_0^{1-x-y} dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^{1-x} xy(1-x-y)^2\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^{1-x} x[y(1-x)^2+y^3-2y^2(1-x$$

$$= \frac{1}{2}\int_0^1 x\left[(1-x)^2\frac{y^2}{2}+\frac{y^4}{4}-\frac{2y^3}{3}(1-x)\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{(1-x)^4}{2}+\frac{(1-x)^4}{4}-\frac{2(1-x)^4}{3}\right] dx$$

$$= \frac{1}{2}\int_0^1 \frac{1}{12}x(1-x)^4 dx = \frac{1}{24}\left[x\frac{(1-x)^5}{5}-(1)\frac{(1-x)^6}{(6)(5)}\right]_0^1$$

$$= \frac{1}{24}\cdot\frac{1}{30} = \frac{1}{720}$$

Example : Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{2}} r\,dz\,dr\,d\theta.$

Solution : Given integral $= \int_0^{\pi/2}\int_0^{a\sin\theta} r\left[\int_0^{(a^2-r^2)/2} dz\right] dr\,d\theta = \int_0^{\pi/2}\int_0^{a\sin\theta} r\left(\frac{a^2-r^2}{2}\right) dr\,d\theta$

$$= \frac{1}{2}\int_0^{\pi/2}\int_0^{a\sin\theta}(a^2r-r^3)\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(\frac{a^2r^2}{2}-\frac{r^4}{4}\right)_0^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\frac{a^2\cdot a^2\sin^2\theta}{2}-\frac{a^4\sin^4\theta}{4}\right)d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2}\sin^2\theta\,d\theta - \frac{a^4}{8}\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$$= \frac{a^4}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{a^4}{8}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi a^4}{16} - \frac{3\pi a^4}{128} = \frac{8\pi a^4 - 3\pi a^4}{128} = \frac{5\pi a^4}{128}$$

Example : Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx.$

Solution : $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx = \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} (e^{z}dz)e^{x}\cdot e^{y}dy\,dx$

$$= \int_0^{\log 2} e^{x}\int_0^{x} e^{y}\,(e^{z})_0^{x+\log y}\,dy\,dx$$

$$= \int_0^{\log 2} e^{x}\int_0^{x} e^{y}\,(e^{x+\log y}-1)\,dy\,dx$$

$$= \int_0^{\log 2} e^{x}\int_0^{x} e^{y}\,(e^{x}\cdot y-1)\,dy\,dx = \int_0^{\log 2} e^{x}\left[\int_0^{x}(ye^{x}-1)e^{y}dy\right]dx$$

$$= \int_0^{\log 2} e^{x}\left[(ye^{x}-1)e^{y}-\int e^{x}\cdot e^{y}dy\right]_0^{x}dx$$

$$= \int_0^{\log 2} e^{x}\left[(ye^{x}-1)e^{y}-e^{x+y}\right]_0^{x}dx$$

$$= \int_0^{\log 2} e^{x}\left[(xe^{x}-1)e^{x}-e^{2x}+1+e^{x}\right]dx$$

$$= \int_0^{\log 2} e^x \left[xe^{2x} - e^x - e^{2x} + 1 + e^x \right] dx$$

$$= \int_0^{\log 2} e^x (xe^{2x} - e^{2x} + 1)\, dx = \int_0^{\log 2} (xe^{3x} - e^{3x} + e^x)\, dx$$

$$= \left[x.\frac{e^{3x}}{3} - \int 1 \cdot \frac{e^{3x}}{3} dx - \frac{e^{3x}}{3} + e^x \right]_0^{\log 2}$$

$$= \left[\frac{x}{3} e^{3x} - \frac{e^{3x}}{9} - \frac{e^{3x}}{3} + e^x \right]_0^{\log 2}$$

$$= \frac{\log 2}{3} e^{3\log 2} - \frac{e^{3\log 2}}{9} - \frac{e^{3\log 2}}{3} + e^{\log 2} + \frac{1}{9} + \frac{1}{3} - 1$$

$$= \frac{8}{3}\log 2 - \frac{8}{9} - \frac{8}{3} + 2 + \frac{1}{9} + \frac{1}{3} - 1 = \frac{8}{3}\log 2 - \frac{19}{9}$$

Example : Evaluate the triple integral $\iiint xy^2 z\,dx\,dy\,dz$ taken through the positive octant of the sphere $x^2+y^2+z^2=a^2$.

Solution : Equation of sphere is $x^2+y^2+z^2=a^2$

The limits of integration are :

$z=0$ to $\sqrt{a^2-x^2-y^2}$, $y=0$ to $\sqrt{a^2-x^2}$ and $x=0$ to a.

$$\iiint xy^2 z\,dx\,dy\,dz = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xy^2 z\,dz\,dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy^2 \left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}} dx\,dy$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} xy^2 (a^2-x^2-y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_0^a x\left[\int_0^{\sqrt{a^2-x^2}} \left\{(a^2-x^2)y^2-y^4\right\} dy\right] dx$$

$$= \frac{1}{2}\int_0^a x\left[\left(\frac{a^2-x^2}{3}\right)y^3 - \frac{y^5}{5}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x\left[\left(\frac{a^2-x^2}{3}\right)(a^2-x^2)^{3/2} - \frac{(a^2-x^2)^{5/2}}{5}\right]dx$$

$$= \frac{1}{2}\int_0^a x(a^2-x^2)^{5/2}\left(\frac{1}{3}-\frac{1}{5}\right)dx$$

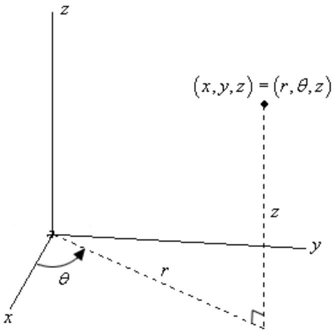
$$= \frac{1}{15}\int_0^a x(a^2-x^2)^{5/2}dx \quad [\text{Put } a^2-x^2=t \Rightarrow -2xdx=dt]$$

$$= \frac{1}{15}\int_{a^2}^{0}\left(-\frac{1}{2}\right)t^{5/2}dt = -\frac{1}{30}\left(\frac{t^{7/2}}{7/2}\right)_{a^2}^{0} = -\frac{1}{30}\left(\frac{2}{7}\right)(-a^7) = \frac{a^7}{105}.$$

Exercise:

1. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dzdydx}{(1+x+y+z)^3}$.
2. Evaluate $\int \underset{V}{\int} \int (2x+y)dxdydz$ where v is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0, x = 2, y = 0, y = 2$ and $z = 0$.

Cylindrical coordinates system



The general limits of r, θ and z are $r \geq 0, \theta \in [0, 2\pi]$ and $z = z$.

Cartesian coordinates to cylindrical coordinates:

The change of cartesian coordinates (x, y, z) to cylindrical coordinates (r, θ, z) by the transformation

$x = r\cos\theta, y = r\sin\theta$ and $z = z$,

then $\int\int_V\int f(x, y, z)dxdydz = \int\int_V\int F(r, \theta, z)|J|drd\theta dz$

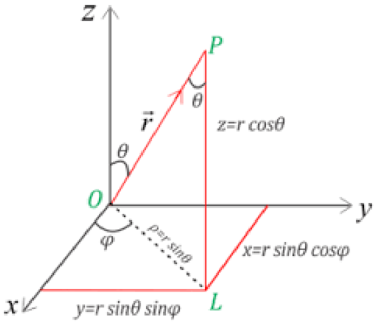
where

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

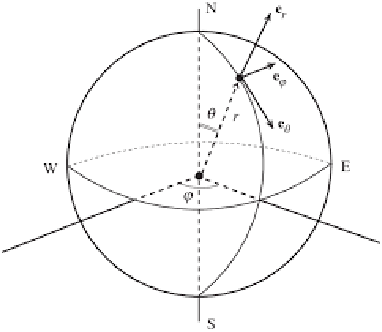
$$\therefore \quad \int\int_V\int f(x, y, z)dxdydz = \int\int_V\int F(r, \theta, z)rdrd\theta dz$$

Spherical coordinates system



The general limits of r, θ *and* ϕ are $r \geq 0, 0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

Spherical coordinates system



Change of variables from cartesian coordinates to spherical coordinates:

Let (x, y, z) denotes cartesian and (r, θ, π) denoted spherical coordinates by the trains formation.
$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta,$
then $\int\int_V\int f(x, y, z)dxdydz = \int\int_V\int F(r, \theta, \phi)|J|drd\theta d\phi$
where

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

$$\therefore \quad \int\int_V\int f(x, y, z)dxdydz = \int\int_V\int F(r, \theta, \phi)r^2\sin\theta drd\theta d\phi.$$

Example 1.7

Evaluate $\int\int\int \frac{dxdydz}{x^2+y^2+z^2}$, taken over the volume bounded by the sphere $x^2+y^2+z^2=a^2$.

Sol. Changing into spherical polar coordinates by putting $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ and $dxdydz = drd\theta d\phi$.
In integrand $x^2+y^2+z^2=r^2$.
Limits: r varies from 0 to a, θ varies from 0 to π, ϕ varies from 0 to 2π.

$$\int\int\int \frac{dxdydz}{x^2+y^2+z^2} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{r^2\sin\theta drd\theta d\phi}{r^2}$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \sin\theta drd\theta d\phi$$
$$= [r]_0^a[\theta]_0^\pi[\phi]_0^{2\pi}$$

Example 1.8

Using cylindrical coordinators, evaluate $\int\int_V\int(x^2+y^2+z^2)dxdydz$ *taken over the region* $0 \le z \le x^2+y^2 \le 1$.

Sol. The region is given by z varies from 0 to 1 and $x^2+y^2=1$.
Changing into cylindrical coordinates by putting $x = r\cos\theta, y = r\sin\theta, z = z$ and $dxdydz = rdrd\theta$.
In the integrand , $x^2+y^2+z^2 = r^2+z^2$.
Limits: z varies from 0 to 1 and the cylinder is above xy-plane and its base is circle. So r varies from 0 to 1 and θ varies from 0 to 2π.

$$\therefore \quad \iiint_V (x^2 + y^2 + z^2)\,dxdydz$$

$$= \int_0^{2\pi} \int_0^1 \int_0^1 (z^2 + r^2)\,rdzdrd\theta$$

$$= \int_0^{2\pi} \int_0^1 \left[\frac{z^3}{3} + r^2 z\right]_0^1 rdrd\theta$$

$$= \int_0^{2\pi} \int_0^1 \left[\frac{1}{3} + r^2\right] rdrd\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{6} + \frac{r^4}{4}\right] d\theta = \frac{5}{12}[\theta]_0^{2\pi} = \frac{5\pi}{6}.$$

Exercise:

1. Changing into spherical polar coordinates, evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdydx$.
   Hint: All the given limits in first(or Positive) octant, so $r : 0 \to a, \theta : 0 \to \frac{\pi}{2}, \phi : 0 \to \frac{\pi}{2}$.
2. Evaluate $\int \underset{V}{\int} \int \frac{dxdydz}{(x^2+y^2+z^2)^{3/2}}$ where v is the region bounded by between two spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$. [Hint: No restrictions like first quadrants. So $r : a \to b, \theta : 0 \to \pi, \phi : 0 \to 2\pi$].
3. Evaluate $\int \underset{V}{\int} \int z(x^2 + y^2)dxdydz$ where v is the volume bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 2$ and $z = 3$ by changing it to cylindrical coordinates. [Hint:$z : 2 \to 3$, and its base is circle, so $r : 0 \to 1, \theta : 0 \to 2\pi$

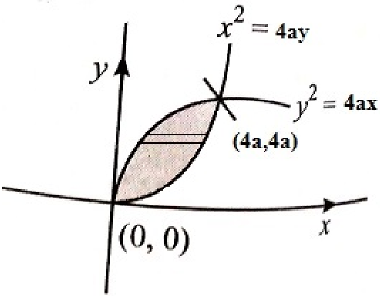
Area enclosed by a curve:

The area of the region bounded by the curves is given by $A = \int\int dxdy$.

Example 1.9

Find the area lying between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Sol. The intersection points are obtained by substituting $x = \frac{y^2}{4a}$ in $x^2 = 4ay$, we get
$\frac{y^4}{16a^2} = 4ay \Rightarrow y^4 = 64a^3y \Rightarrow y(y^3 - 64a^3) = 0$
$\therefore y = 0$ or $y = 4a$.
If $y = 0, x = 0$ and if $y = 4a, x = 4a$.
The points of intersection with the curves are $(0,0)$ and $(4a, 4a)$.
Taking strip parallel to x-axis(or you can select y-axis also), then (from fig) $x : \frac{y^2}{4a}$ to $\sqrt{4ay}$, $y : 0$ to $4a$.



$$\text{Area} = \int\int dxdy = \int_0^{4a}\int_{y^2/4a}^{\sqrt{4ay}} dxdy$$

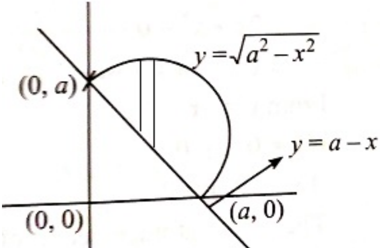
$$= \int_0^{4a} [x]_{y^2/4a}^{\sqrt{4ay}} dy = \int_0^{4a}\left[\sqrt{4ay} - \frac{y^2}{4a}\right] dy$$

$$= \left[\sqrt{4a}\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a} = \frac{2\sqrt{4a}}{3}(4a)^{3/2} - \frac{64}{1\ldots}$$

Example 1.10

Find the area lying between the circle $x^2 + y^2 = a^2$ and the line $x + y = a$ in the first quadrant.

Sol. The intersection points are obtained by substituting $y = a - x$ in $x^2 + y^2 = a^2$, we get
$x^2 + (a - x)^2 = a^2 \Rightarrow 2x^2 - 2ax = 0 \Rightarrow x = 0$ or $x = a$.
If $x = 0, y = a$ and if $x = a, y = 0$. The points of intersection with the curves are $(a, 0)$ and $(0, a)$ which is shown in fig.



Taking strip parallel to y-axis, then $y : a - x$ to $\sqrt{a^2 - x^2}$, $x : 0$ to a.

$$\text{Area} = \int\int dxdy = \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} dydx$$
$$= \int_0^a [y]_{a-x}^{\sqrt{a^2-x^2}} dx = \int_0^a \left[\sqrt{a^2 - x^2} - (a - x$$
$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$
$$= \frac{a^2}{2}.\frac{\pi}{2} - \left(a^2 - \frac{a^2}{2}\right) = \frac{\pi a^2}{4} - \frac{a^2}{2}$$
$$= \frac{a^2}{4}[\pi - 2].$$

Exercise:

1. Find the area bounded by the curves $y = x^3$ and $y = x$.
2. Find the area enclosed by the pair of curves $y = x$ and $y = 4x - x^2$.
3. Find the area of the cardioid $r = a(1 + \cos\theta)$.
   [Hint:$r : 0 \to a(1 + \cos\theta); \theta : 0 \to \pi$.
   Area$= \int\int dxdy = \int\int rdrd\theta$.

Volume as triple integral:

Suppose a solid in three dimension given by planes parallel to the coordinate planes into rectangular parallelepiped *dv* of volume is *dxdydz*.

$\therefore$ The volume of solid is $\int\int_V\int dxdydz$.

Example 1.11

Find the volume under the parabolic $x^2 + y^2 + z = 16$ over the rectangle $x \pm a, y \pm b$.

Sol. Limits: $z : 0$ to $16 - x^2 - y^2$, $y : -b$ to b, $x : -a$ to a.

$$\text{Volume} = \int\int\int dxdydz = \int_{-a}^{a}\int_{-b}^{b}\int_{0}^{16-x^2-y^2} dzdydx$$

$$= \int_{-a}^{a}\int_{-b}^{b}(16 - x^2 - y^2)dydx$$

$$= \int_{-a}^{a}\left[(16 - x^2)y - \frac{y^3}{3}\right]_{-b}^{b}$$

$$= \int_{-a}^{a}\left[2b(16 - x^2) - \frac{2b^3}{3}\right]dx$$

$$= \left[2b\left(16x - \frac{x^3}{3}\right) - \frac{2b^3}{3}x\right]_{-a}^{a}$$

$$= 2b\left[32a - \frac{2a^3}{3}\right] - \frac{4}{3}ab^3 = \frac{4ab}{3}[48 - a^2 - b^2].$$

Example 1.12

Find the volume of the solid cut off from the sphere $x^2+y^2+z^2=a^2$ by the cylinder $x^2+y^2=ax$.

Sol. Given region is $x^2+y^2+z^2=a^2$ and $x^2+y^2=ax$.
Changing into cylindrical coordinates by putting
$x=r\cos\theta, y=r\sin\theta, z=z$ and $dxdydz=rdrd\theta$.
$x^2+y^2+z^2=a^2 \Rightarrow r^2+z^2=a^2$ and
$x^2+y^2=ax \Rightarrow r^2=ar\cos\theta \Rightarrow r=a\cos\theta$.
Limits: we can understand from figure.
$z:-\sqrt{a^2-r^2}\to\sqrt{a^2-r^2}, r:0\to a\cos\theta, \theta:0\to\pi$.

$$\text{Volume}=\int\int\int dxdydz=\int_0^{\pi}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} rdzdrd\theta$$

$$=\int_0^{\pi}\int_0^{a\cos\theta} r[z]_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}}drd\theta$$

$= \int_0^{\pi}\int_0^{a\cos\theta} 2r\sqrt{a^2-r^2}\,dr\,d\theta$ Put $a^2 - r^2 = t^2$
$-2rdr = 2tdt$. Limits: $r \to a\cos\theta \Rightarrow t \to a\sin\theta$ and
$r \to 0 \Rightarrow t \to a$

$$\therefore \quad \text{Volume} = -\int_0^{\pi}\int_a^{a\sin\theta} t(2t)\,dt\,d\theta$$
$$= -\int_0^{\pi}\left[\frac{2t^3}{3}\right]_a^{a\sin\theta} d\theta$$
$$= -\int_0^{\pi}\left[\frac{2a^3}{3}\sin^3\theta - \frac{2a^3}{3}\right] d\theta$$
$$= -\frac{2a^3}{3}\int_0^{\pi}\left[\frac{3\sin\theta - \sin 3\theta}{4} - 1\right] d\theta$$
$$= -\frac{2a^3}{3}\left[\frac{-3}{4}\cos\theta + \frac{1}{12}\cos 3\theta - \theta\right]_0^{\pi}$$
$$= -\frac{2a^3}{3}\left[\frac{6}{4} - \frac{2}{12} - \pi\right]$$

Exercise:

1. Find the volume of tetra hedron bounded by the planes $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
   Hint: using triple integration.
2. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ using triple integration.
   Hint: Also use spherical polar coordinates.
3. Find the volume of the solid enclosed by the surface $z = 0; x^2 + y^2 = cz$ and $x^2 + y^2 = 2ax$.
   Hint: Using cylindrical coordinates:
   $z : 0 \to \frac{r^2}{c}, r : 0 \to 2a\cos\theta, \theta : 0 \to \pi$ because entire cylinder exists in right side.

Multiple Integrals & applications of integration

1. $\iint x^2y^2dxdy$ over the circle $x^2+y^2=1$ is

a) Π/2 b) Π/24 c) Π/36 d)None

2. $\int_0^1\int_0^{x^2} e^{y/x}dydx=$

a)1 b)1/2 c)1/4 d)None

3. $\int_{-1}^{2}\int_{x^2}^{x+2} dydx=$

a)9/2 b)9/4 c)3/2 d)None

4. $\int_0^1\int_0^1\int_0^1 e^{x+y+z}dxdydz=$

a)$(e-1)^2$ b)e^{-1} c)(e-1)/2 d)$e^{-1/2}$

5. The limits of integration after the change of order of integration of $\int_0^2\int_1^{x^2} f(x,y)dydx$

a)x=3 $y^{1/2}$ to 2,y=0 to 8 b)x=y^3 to 2 ,y=0 to 2 c)x=0 to y^3, y=0 to 2 d)None

6. $\iint x^2y^3dxdy$ over the rectangle 0<=x<=1 and 0<=y<=3 is

a) 81/4 b)27/8 c)29/4 d)None

7. $\iiint (z^5 + z)dxdydz$ over the volume of cube bounded by the planes x=y=z=9 is

a)0 b)4 Π c)8 Π/3 d)None

8. $\iint ydxdy$ over the area bounded by x=0,y=x^2,x+y=2 in the first quadrant is

a)8/15 b)16/15 c)4/15 d)None

9.The value of the double integral $\int_0^2\int_0^2 (4-y^2)dydx$

a)16 b)16/3 c)8/3 d)None

10. $\iint_A dA$, where A is the region in first quadrant bounded by the lines y=x,y=2x,x=1 and x=2is

a)3 b)3/2 c)3/8 d)None

11.The length of the curve y=$x^{3/2}$ from x=0to x=4/3 is

a)26/27 b)36/27 c)46/27 d)56/27

12.The length of the arc x=t,y=t from t=0 to t=4 is

a) $\sqrt{2}$ b) $2^{3/2}$ c) $3\sqrt{2}$ d) $4\sqrt{2}$

13.The length of the arc of the curve $x= e^t \sin t$, $y= e^t \cos t$ from t=0 to t= Π/2

a) $2e^{\Pi/2}$ b) $e^{\Pi/2}-1$ c) $2(e^{\Pi/2}-1)$ d) $\sqrt{2}(e^{\Pi/2}-1)$

14.The perimeter of the loop of the curve $3ay^2=x(x-a)^2$ is

a)2a/3 b)4a/3 c) $4a/\sqrt{3}$ d) $2a/\sqrt{3}$

15.The perimeter of the curve r=a(sin t+cos t), 0<=t<= Π is

a)2a Π b) $\sqrt{2}$ a Π c)3a Π d) $\sqrt{3}$ aΠ

16. The volume of a cone of height h and base radius r is

a) $4/3\ \Pi r^2 h$ b) $2/3\ \Pi r^2 h$ c) $1/3\ \Pi r^2 h$ d) $\Pi r^2 h$

17.The volume of the paraboloid generated by revolving the parabola y2= 4ax about the x-axis from x=0 to x=h

a) $\Pi a h^2$ b) $2\ \Pi a h^2$ c) $3\ \Pi a h^2$ d) $4\ \Pi a h^2$

18.The volume of the solid formed by revolving the ellipse $x^2/a^2+y^2/b^2=1$ about the major axis is

a)$2/3\ \Pi a^2b$ b)$2/3\ \Pi ab^2$ c)$4/3\ \Pi a^2b$ d)$4/3\ \Pi ab^2$

19.The volume of the solid generated by revolving the cardioid r =a(1+cos t),0<=t<= Π is

a) $\Pi/3a^3$ b)$2\ \Pi/3\ a^3$ c)$4\ \Pi/3\ a^3$ d)$8\ \Pi/3\ a^3$

20. The volume of the solid generated by revolving the area enclosed by y=x,y=0 and x=a about the x-axis is

a) Πa^3 b)$2\ \Pi a^3$ c)$2/3\ \Pi a^3$ d) $\Pi/3a^3$

Key : 1b 2.b 3.a 4. c 5.a 6.d 7.a 8.b 9.d 10.b

11.d 12.d 13.d 14. c 15.b 16.c 17.b 18.c 19.b 20.d