ORDINARY DIFFERENTIAL EQUATIONS:

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Variable-separable-method:

1. Solve $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

Solution. We have,

$$\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$$

$$\Rightarrow (\sin y + y\cos y)dy = \{x(2\log x+1)\}dx$$

On integrating both the sides, we get

$$\int(\sin y + y\cos y)dy = \int\{x(2\log x+1)\}dx + C$$

$$-\cos y + y\sin y - \int(1)\cdot\sin y\,dy = 2\int\log x\cdot x\,dx + \int x\,dx + C$$

$$\Rightarrow -\cos y + y\sin y + \cos y$$

$$= 2\left[\log x\cdot\frac{x^2}{2} - \int\frac{1}{x}\cdot\frac{x^2}{2}dx\right] + \frac{x^2}{2} + C$$

$$y \sin y = 2\log x \cdot \frac{x^2}{2} - \int x\,dx + \frac{x^2}{2} + C$$

$$y \sin y = 2\log x \cdot \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{2} + C$$

$$y \sin y = x^2 \log x + C$$

2. Solve

(i) $(x+1)\dfrac{dy}{dx} = x(y^2+1)$

(ii) $(xy^2+x)dx + (yx^2+y)dy = 0$

Solution. (i) We have $(x+1)\dfrac{dy}{dx} = x(y^2+1)$

$$\Rightarrow \quad (x+1)dy = x(y^2+1)\,dx$$

$$\Rightarrow \quad \frac{dy}{y^2+1} = \frac{xdx}{x+1}$$

$$\Rightarrow \quad \frac{1}{1+y^2}dy = \left(1-\frac{1}{x+1}\right)dx$$

On integrating, we get $\displaystyle\int \frac{1}{1+y^2}dy = \int dx - \int \frac{1}{x+1}dx + C$

$$\Rightarrow \quad \tan^{-1} y = x - \log(1+x) + C$$

(ii) We have $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0$

$$\Rightarrow \quad (y^2+1)\,x\,dx = -(x^2+1)\,y\,dy$$

$$\Rightarrow \quad \frac{y\,dy}{y^2+1} = -\frac{x}{x^2+1}dx$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{2y}{y^2+1}dy = -\frac{1}{2}\int \frac{2x}{x^2+1}dx + \frac{1}{2}C$$

$$\Rightarrow \quad \frac{1}{2}\log(y^2+1) = -\frac{1}{2}\log(x^2+1) + \frac{1}{2}C$$

$$\Rightarrow \quad \log(y^2+1) + \log(x^2+1) = \log C'$$

$$\Rightarrow \quad (y^2+1)(x^2+1) = C'$$

4. Find the solution of the differential equation

$$\frac{dy}{dx} - x\tan(y-x) = 1$$

Solution. We have $\frac{dy}{dx} - x\tan(y-x) = 1$

On putting, $y - x = t$ so that $\frac{dy}{dx} - 1 = \frac{dt}{dx}$ in Eq. (i), we get

$$\left(1 + \frac{dt}{dx}\right) - x\tan t = 1, \quad \frac{dt}{dx} = x\tan t \quad \Rightarrow \quad \cot t \, dt = x\,dx$$

On integrating, we have $\log \sin t = \frac{x^2}{2} + C$

$$\Rightarrow \qquad \log \sin(y-x) = \frac{x^2}{2} + C$$

Linear-equation:

5.

Solve $x(x-1)\frac{dy}{dx} - (x-2)y = x^2(2x-1)$

Solution. We have $\frac{dy}{dx} - \frac{x-2}{x(x-1)}y = \frac{x^2(2x-1)}{x(x-1)}$

Hence, the integrating factor is

$$IF = e^{-\int \frac{x-2}{x(x-1)}dx} = e^{\int \left(\frac{1}{x-1} - \frac{2}{x}\right)dx}$$

$$= e^{\log(x-1) - 2\log x} = e^{\log\frac{x-1}{x^2}} = \frac{x-1}{x^2}$$

Hence, the solution is

$$y \cdot \frac{x-1}{x^2} = \int \frac{x^2(2x-1)}{x(x-1)} \cdot \frac{x-1}{x^2} dx + C = \int \frac{2x-1}{x} dx + C$$

$$= \int \left(2 - \frac{1}{x}\right) dx + C = 2x - \log x + C$$

6. *If* $\frac{dy}{dx} + 2y \tan x = \sin x$ *and* $y\left(\frac{\pi}{3}\right) = 0$, *then find maximum value of* $y(x)$.

Solution. We have $\frac{dy}{dx} + 2y\tan x = \sin x$

$$IF = e^{2\int \tan x\,dx} = e^{2\log\sec x} = e^{\log\sec^2 x} = \sec^2 x$$

Hence, solution is

$$y \cdot \sec^2 x = \int \sec^2 x \cdot \sin x\,dx + C = \int \tan x \sec x\,dx + C$$

$$= \sec x + C$$

Put $x = \frac{\pi}{3}, y = 0$ in Eq. (i),

$$0 \times \sec^2\frac{\pi}{3} = \sec\frac{\pi}{3} + C$$

$$0 = 2 + C \text{ or } C = -2$$

Substituting $C = -2$ in Eq. (i), we get

$$y\sec^2 x = \sec x - 2$$

$$y = \cos x - 2\cos^2 x,$$

$$\frac{dy}{dx} = -\sin x + 4\cos x \sin x$$

For maxima or minima, we have

$$0 = -\sin x + 4\sin x\cos x$$

$$\Rightarrow \quad -\sin x(1 - 4\cos x) = 0$$

$$\Rightarrow \quad \sin x = 0 \text{ or } \cos x = \frac{1}{4}$$

Now, $$\frac{d^2y}{dx^2} = -\cos x - 4\sin^2 x + 4\cos^2 x$$

If $\cos x = \frac{1}{4}$, then

$$\frac{d^2y}{dx^2} = -\frac{1}{4} - 4\sqrt{1 - \frac{1}{16}} + 4\left(\frac{1}{16}\right) = -\sqrt{15}$$

So, y is maximum if $\cos x = \frac{1}{4}$, Maximum value of

$$y = \frac{1}{4} - \frac{2}{16} = \frac{1}{8}$$

7. Solve $x^2 dy + y(x+y)dx = 0$

Solution. We have, $x^2 dy + y(x+y)dx = 0$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = -\frac{y^2}{x^2} \quad \Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = -\frac{1}{x^2}$$

Put $-\frac{1}{y} = z$ so that $\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

The given equation reduces to a linear differential equation in z, as

$$\frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$$

Hence, $IF = e^{-\int 1/x\, dx} = e^{-\log x} = e^{\log 1/x} = \frac{1}{x}$

Hence, the solution is

$$z \cdot \frac{1}{x} = \int -\frac{1}{x^2} \cdot \frac{1}{x} dx + C \quad \Rightarrow \quad \frac{z}{x} = \int -x^{-3} dx + C$$

$$\Rightarrow \quad -\frac{1}{xy} = -\frac{x^{-2}}{-2} + C \quad \Rightarrow \quad \frac{1}{xy} = -\frac{1}{2x^2} - C$$

8. *Solve* $(\sec x \tan x \tan y - e^x)dx + \sec x \sec^2 y\, dy = 0$

Solution. We have $\sec x \tan x \tan y - e^x + \sec x \sec^2 y \dfrac{dy}{dx} = 0$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dx} + \tan x \tan y = e^x \cos x$$

Put $\tan y = z, \Rightarrow \sec^2 y \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$$\Rightarrow \quad \frac{dz}{dx} + z \tan x = e^x \cos x$$

$$IF = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

Hence, the solution is $z \sec x = \int e^x \cos x \sec x\, dx + C$

$$\Rightarrow \quad z \sec x = \int e^x dx + C$$

$$\Rightarrow \quad \tan y \sec x = e^x + C$$

9. Solve the differential equation :

$$\frac{dy}{dx}+\frac{2x}{(1+x^2)}y=\frac{1}{(1+x^2)^2} \text{ given } y=0, \text{ when } x=1.$$

Solution : Given differential equation is

$$\frac{dy}{dx}+\frac{2x}{(1+x^2)}y=\frac{1}{(1+x^2)^2} \qquad \ldots(1)$$

This is a linear equation in y.

Here $P=\dfrac{2x}{1+x^2}$; $Q=\dfrac{1}{(1+x^2)^2}$.

$$\text{I.F.} = e^{\int P dx} = e^{\int\left(\frac{2x}{1+x^2}\right)dx} = e^{\log(1+x^2)} = 1+x^2$$

$\therefore$ The general solution is

$$y(1+x^2)=\int\frac{1}{(1+x^2)^2}\cdot(1+x^2)dx+c=\int\frac{1}{1+x^2}dx+c$$

i.e. $y(1+x^2) = \tan^{-1} x + c$ ... (2)

Given condition is $y = 0$, when $x = 1$.

$$\Rightarrow \quad 0 = \tan^{-1}(1) + c \quad \text{or} \quad c = -\frac{\pi}{4}$$

Substituting 'c' value in equation (2), we get

$$y(1+x^2) = \tan^{-1} x - \frac{\pi}{4}.$$

Exact and Non-exact:

10. *Solve* $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}$

Solution. Given, differential equation can be written as

$$\left(x + \frac{a^2y}{x^2+y^2}\right)dx + \left(y - \frac{a^2x}{x^2+y^2}\right)dy = 0$$

Here, $M = x + \dfrac{a^2y}{x^2+y^2}$

and $N = y - \dfrac{a^2x}{x^2+y^2}$

Now, $\dfrac{\partial M}{\partial y} = \dfrac{a^2(x^2-y^2)}{(x^2+y^2)^2} = \dfrac{\partial N}{\partial x}$

Hence, given, differential equation is exact. Hence, required solution is

$$\underbrace{\int\left(x+\frac{a^2y}{x^2+y^2}\right)dx}_{y\,=\,\text{constant}}+\underbrace{\int\left(y-\frac{a^2x}{x^2+y^2}\right)dy}_{\text{Terms free from }x}=C$$

$$\Rightarrow \quad \int x\,dx+a^2y\int\frac{dx}{x^2+y^2}+\int y\,dy=C$$

$$\Rightarrow \quad \frac{x^2}{2}+a^2y\cdot\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)+\frac{y^2}{2}=C$$

$$\Rightarrow \quad \left(\frac{x^2+y^2}{2}\right)+a^2\tan^{-1}\left(\frac{x}{y}\right)=C$$

11. Solve $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

Solution : Given equation is $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$ ...(1)

This is of the form $Mdx+Ndy=0$ where

$$M=1+e^{\frac{x}{y}},\ N=e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$$

$$\therefore\quad \frac{\partial M}{\partial y}=e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right)$$

and $\frac{\partial N}{\partial x}=e^{\frac{x}{y}}\left(\frac{-1}{y}\right)+\left(1-\frac{x}{y}\right)e^{\frac{x}{y}}\left(\frac{1}{y}\right)$

$$=e^{\frac{x}{y}}\left(-\frac{1}{y}+\frac{1}{y}-\frac{x}{y^2}\right)=e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right)$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, therefore, the given equation is exact.

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore, the given equation is exact.

Hence the general solution is given by

$$\int_{(y \text{ constant})} M dx + \int (\text{terms independent of } x \text{ in } N) dy = c$$

$$\Rightarrow \int_{(y \text{ constant})} \left(1 + e^{x/y}\right) dx + \int 0 dy = c \Rightarrow x + \frac{e^{\frac{x}{y}}}{1/y} = c \Rightarrow x + ye^{\frac{x}{y}} = c$$

12. Solve $(2x\log x - xy)dy + 2y dx = 0$

Solution. Here, $M = 2y$ and $N = 2x\log x - xy$

Hence, $\frac{\partial M}{\partial y} = 2$ and $\frac{\partial N}{\partial x} = 2(1+\log x) - y$

Here, $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{-2\log x + y}{2x\log x - xy} = -\dfrac{1}{x} = f(x)$

Now, $IF = e^{\int f(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \dfrac{1}{x}$

On multiplying the given differential equation by $\frac{1}{x}$, we get

$$\frac{2y}{x}dx + (2\log x - y)dy = 0 \Rightarrow \int \frac{2y}{x}dx + \int -y dy = C$$

$$\Rightarrow \quad 2y\log x - \frac{1}{2}y^2 = C$$

13. Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

Solution. Here, $M = y^4 + 2y$ and $N = xy^3 + 2y^4 - 4x$

$$\therefore \quad \frac{\partial M}{\partial y} = 4y^3 + 2 \text{ and } \frac{\partial N}{\partial x} = y^3 - 4$$

$$\therefore \quad \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{(y^3 - 4) - (4y^3 + 2)}{y^4 + 2y} = \frac{-3(y^3 + 2)}{y(y^3 + 2)} = -\frac{3}{y} = f(y)$$

$$IF = e^{\int f(y)dy} = e^{\int -\frac{3}{y}dy} = e^{-3\log y}$$

$$= e^{\log y^{-3}} = y^{-3} = \frac{1}{y^3}$$

On multiplying the given Eq. (i) by $\frac{1}{y^3}$, we get the exact differential equation as

$$\left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)dy=0$$

$$\Rightarrow \quad \int\left(y+\frac{2}{y^2}\right)dx+\int 2y\,dy=C$$

$$\Rightarrow \quad x\left(y+\frac{2}{y^2}\right)+y^2=C$$

14. Solve $y^2dx+\left(x^2-xy-y^2\right)dy=0$

Solution : The given differential equation is $y^2dx+(x^2-xy-y^2)dy=0$ ...(1)

This is of the form $Mdx+Ndy=0$ where $M=y^2,\ N=x^2-xy-y^2$

We have $\dfrac{\partial M}{\partial y}=2y$ and $\dfrac{\partial N}{\partial x}=2x-y$ so that $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$

$\therefore$ The given equation is non-exact . But it is a homogeneous equation.

Thus I.F. $=\dfrac{1}{Mx+Ny}=\dfrac{1}{y\left(x^2-y^2\right)}$.

Multiplying (1) with $\dfrac{1}{y(x^2-y^2)}$, we get

$$\frac{y}{x^2-y^2}dx+\frac{x^2-xy-y^2}{y(x^2-y^2)}dy=0 \qquad \text{...(2)}$$

This is of the form $M_1=\dfrac{y}{x^2-y^2}$ and $N_1=\dfrac{x^2-xy-y^2}{y(x^2-y^2)}$

We have $\frac{\partial M_1}{\partial y} = \frac{(x^2 - y^2)(1) - y(-2y)}{(x^2 - y^2)^2} = \frac{x^2 - y^2 + 2y^2}{(x^2 - y^2)^2} = \frac{x^2 + y^2}{(x^2 - y^2)^2}$

and $\frac{\partial N_1}{\partial x} = \frac{1}{y}\left[\frac{(x^2 - y^2)(2x - y) - (x^2 - xy - y^2)(2x)}{(x^2 - y^2)^2}\right] = \frac{1}{y}\left[\frac{y^3 + x^2 y}{(x^2 - y^2)^2}\right] = \frac{x^2 + y^2}{(x^2 - y^2)^2}$

$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ so that equation (2) is exact.

General solution of (2) is given by

$$\int_{(y\text{ constant})} M_1 dx + \int (\text{terms independent of } x \text{ in } N_1) dy = c$$

$$\Rightarrow \int_{(y\text{ constant})} \frac{y}{x^2 - y^2} dx + \int \frac{1}{y} dy = c \Rightarrow y \int_{(y\text{ constant})} \frac{1}{x^2 - y^2} dx + \log|y| = c$$

$$\Rightarrow y.\frac{1}{2y} \log\left|\frac{x-y}{x+y}\right| + \log|y| = \log c \Rightarrow \log\left(\frac{x-y}{x+y}\right)^{1/2} + \log|y| = \log c$$

$$\Rightarrow \left(\frac{x-y}{x+y}\right)^{1/2} y = c$$

$$\Rightarrow (x-y)y^2 = c^2(x+y).$$

This is the general solution of (2) and hence of (1).

15. *Solve* $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

Solution. We have $(x^2+y^2)dx-(xy)dy=0$ ...(i)

Hence, $M = x^2+y^2$ and $N=-xy$

$$Mx+Ny = x^3+xy^2-xy^2 = x^3 \neq 0$$

Hence, $IF = \frac{1}{x^3}$

Multiplying Eq. (i) both sides by $\frac{1}{x^3}$, we get

$$\left(\frac{1}{x}+\frac{y^2}{x^3}\right)dx-\left(\frac{y}{x^2}\right)dy=0$$

which is an exact differential equation.

Hence, Solution is $\underbrace{\int\left(\frac{1}{x}+\frac{y^2}{x^3}\right)d}_{y\,=\,\text{constant}} - \underbrace{\int\frac{y}{x^2}dy}_{\text{Terms free from } x} = C$

y = constant   Terms free from x

$$\int \frac{dx}{x} + y^2 \int \frac{dx}{x^3} - \int 0 \cdot dy = C$$

$$\log x - \frac{y^2}{2x^2} = C$$

16. Solve $y(1+xy)dx + x(1-xy)dy = 0$

Solution : Given D. E. is of the form $Mdx + Ndy = 0$ so that $M = y(1+xy)$ and $N = x(1-xy)$

$$\therefore \frac{\partial M}{\partial y} = 1 + 2xy \text{ and } \frac{\partial N}{\partial x} = 1 - 2xy$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact D. E.

The given D. E. is of the form $y.f(xy) + x.g(xy)\, dy = 0$

Now $Mx - Ny = xy\,(1+xy) - xy\,(1-xy)$

$$= xy[(1+xy)-(1-xy)] = 2x^2y^2 \neq 0$$

$\therefore$ I. F. $= \dfrac{1}{Mx - Ny} = \dfrac{1}{2x^2y^2}$

Multiplying the given D. E. with I. F. *i.e.*, $\dfrac{1}{2x^2y^2}$, we get

$$\frac{1}{2x^2y}(1+xy)\,dx + \frac{1}{2xy^2}(1-xy)\,dy = 0$$

i.e., $\left(\dfrac{1}{2x^2y} + \dfrac{1}{2x}\right)dx + \left(\dfrac{1}{2xy^2} - \dfrac{1}{2y}\right)dy = 0$ ...(1)

This is of the form $M_1 dx + N_1 dy = 0$

Here $\dfrac{\partial M_1}{\partial y} = \dfrac{-1}{2x^2y^2}$ and $\dfrac{\partial N_1}{\partial x} = \dfrac{-1}{2x^2y^2}$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, the equation (1) is an Exact D. E.

$\therefore$ The general solution of (1) is

$$\underset{(y\,\text{constant})}{\int M_1 dx} + \underset{(\text{terms not containing } x)}{\int N_1 dy} = c$$

i.e., $$\int_{(y\text{ constant})} \left(\frac{1}{2x^2 y} + \frac{1}{2x} \right) dx + \int \frac{-1}{2y} dy = c$$

i.e., $$\frac{1}{2y}\left(\frac{x^{-2+1}}{-2+1} \right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

i.e., $$-\frac{1}{2xy} + \frac{1}{2}\log\left(\frac{x}{y}\right) = c$$

or $$\log\left(\frac{x}{y}\right) - \frac{1}{xy} = c$$

17. Solve $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0$

Solution. Here, $M = y^3 - 2x^2y$ and $N = 2xy^2 - x^3$

$$\frac{\partial M}{\partial y} = 3y^2 - 2x^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 2y^2 - 3x^2$$

Clearly, $$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Multiplying both sides the above differential equation by, $x^\alpha y^\beta$, so that it reduces in the form of $M_1dx + N_1dy = 0$

where, $$M_1 = x^\alpha y^{\beta+3} - 2x^{\alpha+2}y^{\beta+1},$$

and $$N_1 = 2x^{\alpha+1}y^{\beta+2} - x^{\alpha+3}y^\beta$$

Now, $$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$$\Rightarrow \quad x^\alpha \cdot (\beta + 3)y^{\beta+2} - 2x^{\alpha+2}(\beta + 1)y^\beta$$
$$= 2y^{\beta+2} \cdot (\alpha + 1)x^\alpha - y^\beta \cdot (\alpha + 3)x^{\alpha+2}$$

18. Solve $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0$

Solution. Here, $M = y^3 - 2x^2y$ and $N = 2xy^2 - x^3$

$$\frac{\partial M}{\partial y} = 3y^2 - 2x^2 \text{ and } \frac{\partial N}{\partial x} = 2y^2 - 3x^2$$

Clearly, $$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

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where, $$M_1 = x^\alpha y^{\beta+3} - 2x^{\alpha+2}y^{\beta+1},$$

and $$N_1 = 2x^{\alpha+1}y^{\beta+2} - x^{\alpha+3}y^\beta$$

Now, $$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$$\Rightarrow x^\alpha \cdot (\beta+3)y^{\beta+2} - 2x^{\alpha+2}(\beta+1)y^\beta$$
$$= 2y^{\beta+2} \cdot (\alpha+1)x^\alpha - y^\beta \cdot (\alpha+3)x^{\alpha+2}$$

Dividing both sides by $x^\alpha y^\beta$, we get

$$\Rightarrow (\beta+3)y^2 - 2(\beta+1)x^2 = 2(\alpha+1)y^2 - (\alpha+3)x^2$$

$$\Rightarrow \quad (\alpha+3-2\beta-2)x^2+(\beta+3-2\alpha-2)y^2=0$$

$$\Rightarrow \quad \alpha-2\beta+1=0, \quad 2\alpha-\beta-1=0$$

$$\Rightarrow \quad \alpha=1, \quad \beta=1$$

Hence, $\quad IF = xy$

Hence, required solution is

$$\underbrace{\int M_1 dx}_{y\,=\,\text{constant}} + \underbrace{\int N_1 dy}_{\text{Terms free from } x} = C$$

$$\Rightarrow \quad \int (xy^4-2x^3y^2)dx + \int (2x^2y^3 - x^4y)dy = C$$

y = constant $\qquad$ Terms free from x

$$\Rightarrow \quad y^4\int x dx - 2y^2\int x^3 dx + \int 0\cdot dy = C$$

$$\Rightarrow \quad \frac{x^2y^4}{2} - \frac{x^4y^2}{2} = C$$

$$\Rightarrow \quad x^2y^4 - x^4y^2 = C'$$

Solve $(d^4y/dx^4) - (d^3y/dx^3) - 9(d^2y/dx^2) - 11(dy/dx) - 4y = 0$.

Sol. Let $D = d/dx$. Then the given equation can be written as

$(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$ or $(D + 1)^3 (D - 4) = 0$ so that $D = 4, -1, -1, -1$.

The required solution is $y = c_1 e^{4x} + (c_2 + c_3 x + c_4 x^2) e^{-x}$, c_1, c_2, c_3, c_4 being arbitrary constants.

Solve $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$.

Sol. A.E. is $D^3 - 6D^2 + 11D - 6 = 0$ or $(D-1)(D-2)(D-3) = 0$

whence, $D = 1, 2, 3$

Therefore, $\text{C.F.} = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$

$$\text{P.I.} = \frac{1}{D^3 - 6D^2 + 11D - 6}(e^{-2x} + e^{-3x})$$

$$= \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-2x} + \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-3x}$$

$$= \frac{1}{(-2)^3 - 6(-2)^2 + 11(-2) - 6} e^{-2x} + \frac{1}{(-3)^3 - 6(-3)^2 + 11(-3) - 6} e^{-3x}$$

$$= -\frac{1}{60} e^{-2x} - \frac{1}{120} e^{-3x} = -\frac{1}{120}(2e^{-2x} + e^{-3x})$$

Hence the C.S. is $y = \text{C.F.} + \text{P.I.}$

Find the P.I. of $(D^3 + 1)y = \sin(2x + 3)$.

Sol.

$$\text{P.I.} = \frac{1}{D^3 + 1}\sin(2x+3) = \frac{1}{D(-2^2)+1}\sin(2x+3)$$

[Putting $D^2 = -2^2$]

$$= \frac{1}{1-4D}\sin(2x+3)$$

Multiplying and dividing by $(1 + 4D)$

$$= \frac{1+4D}{(1-4D)(1+4D)}\sin(2x+3) = \frac{1+4D}{1-16D^2}\sin(2x+3)$$

$$= \frac{1+4D}{1-16(-2^2)}\sin(2x+3)$$

[Putting $D^2 = -2^2$]

$$= \frac{1}{65}[\sin(2x+3) + 4D\sin(2x+3)]$$

$$= \frac{1}{65}[\sin(2x+3) + 8\cos(2x+3)]$$

$\left[\text{Since } D = \frac{d}{dx}\right]$

Find the P.I. of $(D^2 + 4)y = \cos 2x$.

Sol. $$\text{P.I.} = \frac{1}{D^2 + 4} \cos 2x$$

Here the denominator vanishes when D^2 is replaced by $-2^2 = -4$. It is a case of failure. multiply the numerator by x and differentiate the denominator w.r.t. D.

$$\therefore \quad \text{P.I.} = x \cdot \frac{1}{2D} \cos 2x = \frac{x}{2} \int \cos 2x \, dx \qquad \left[\text{Since } \frac{1}{D} f(x) = \int f(x) dx\right]$$

$$= \frac{x}{4} \sin 2x.$$

Find the P.I. of $(D^2 + 5D + 4)y = x^2 + 7x + 9.$

Sol.

$$\text{P.I.} = \frac{1}{D^2+5D+4}(x^2+7x+9) = \frac{1}{4\left(1+\frac{5D}{4}+\frac{D^2}{4}\right)}(x^2+7x+9)$$

$$= \frac{1}{4}\left[1+\left(\frac{5D}{4}+\frac{D^2}{4}\right)\right]^{-1}(x^2+7x+9)$$

$$= \frac{1}{4}\left[1-\left(\frac{5D}{4}+\frac{D^2}{4}\right)+\left(\frac{5D}{4}+\frac{D^2}{4}\right)^2-\ldots\ldots\right](x^2+7x+9)$$

$$= \frac{1}{4}\left(1-\frac{5D}{4}-\frac{D^2}{4}+\frac{25D^2}{16}\ldots\ldots\right)(x^2+7x+9)$$

$$= \frac{1}{4}\left(1-\frac{5D}{4}+\frac{21D^2}{16}\ldots\ldots\right)(x^2+7x+9)$$

$$= \frac{1}{4}\left[(x^2+7x+9)-\frac{5}{4}D(x^2+7x+9)+\frac{21}{16}D^2(x^2+7x+9)\right]$$

$$= \frac{1}{4}\left[(x^2+7x+9)-\frac{5}{4}(2x+7)+\frac{21}{16}(2)\right] = \frac{1}{4}\left(x^2+\frac{9}{2}x+\frac{23}{8}\right).$$

Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$.

Sol. A.E. is $(D+2)(D-1)^2 = 0$ so that $D = -2, 1, 1$

Therefore, $\quad$ C.F. $= c_1 e^{-2x} + (c_2 + c_3 x)\, e^x$

$$\text{P.I.} = \frac{1}{(D+2)(D-1)^2}\,(e^{-2x} + 2\sinh x)$$

$$= \frac{1}{(D+2)(D-1)^2}\,(e^{-2x} + e^x - e^{-x}) \qquad \left[\text{Since} \quad \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

Now, $$\frac{1}{(D+2)(D-1)^2}\, e^{-2x} = \frac{1}{D+2}\left[\frac{1}{(D-1)^2}\, e^{-2x}\right] = \frac{1}{D+2}\left[\frac{1}{(-2-1)^2}\, e^{-2x}\right]$$

$$= \frac{1}{9}\cdot\frac{1}{D+2}\, e^{-2x} \qquad | \text{ Case of failure}$$

$$= \frac{1}{9}x\,.\,\frac{1}{1}e^{-2x} = \frac{x}{9}e^{-2x}$$

Solve $(D^2 + 1)\, y = \sin x \sin 2x + e^x x^2$.

Sol. Given differential equation is $(D^2 + 1) = \sin x \sin 2x + e^x . x^2$

A.E. is $D^2 + 1 = 0 \;\Rightarrow\; D = \pm i$

$$\text{C.F. } y = c_1 \cos x + c_2 \sin x. \qquad \ldots(1)$$

$$\text{P.I.} = \frac{1}{D^2+1}(\sin x \sin 2x + e^x x^2) = \frac{1}{D^2+1} \sin x \sin 2x + \frac{1}{D^2+1} x^2 . e^x$$

Consider $\frac{1}{D^2+1} \sin x \sin 2x = \frac{1}{D^2+1}\left(\frac{\cos x - \cos 3x}{2}\right)$

$$= \frac{1}{2}\left[\frac{1}{D^2+1}\cos x - \frac{1}{D^2+1}\cos 3x\right] = \frac{1}{2}\left[x \sin x + \frac{1}{8}\cos 3x\right]$$

Next, $\frac{1}{D^2+1} x^2 e^x = e^x \frac{1}{(D+1)^2+1} x^2 = e^x \frac{1}{D^2+2D+2} x^2$

$$= e^x \frac{1}{2}\left(\frac{1}{1+\frac{D^2+2D}{2}} x^2\right) = \frac{e^x}{2}\left[1 - \frac{D^2+2D}{2} + \left(\frac{D^2+2D}{2}\right)^2 - \ldots\ldots\right] x^2$$

$$= \frac{e^x}{2}\left[x^2 - \left(\frac{1+2x}{2}\right) + \frac{1}{4}(8)\right] = \frac{e^x}{2}\left[x^2 - 2x + \frac{3}{2}\right]$$

So $$\text{P.I. } y = \frac{x \sin x}{2} + \frac{\cos 3x}{16} + \frac{e^x}{2}\left(x^2 - x + \frac{3}{2}\right) \qquad \ldots(2)$$

Using (1) and (2), the complete solution is

Find the P.I. of $(D^2 - 4D + 3)y = e^x \cos 2x$.

$$\text{P.I.} = \frac{1}{D^2 - 4D + 3} e^x \cos 2x = e^x \frac{1}{(D+1)^2 - 4(D+1) + 3} \cos 2x$$

$$= e^x \frac{1}{D^2 - 2D} \cos 2x = e^x \frac{1}{-2^2 - 2D} \cos 2x \qquad [\text{Putting } D^2 = -2^2]$$

$$= -\frac{1}{2} e^x \frac{1}{2+D} \cos 2x = -\frac{1}{2} e^x \frac{2-D}{(2+D)(2-D)} \cos 2x$$

$$= -\frac{1}{2} e^x \frac{2-D}{4-D^2} \cos 2x = -\frac{1}{2} e^x \frac{2-D}{4-(-2^2)} \cos 2x$$

$$= -\frac{1}{16} e^x (2\cos 2x - D\cos 2x) = -\frac{1}{16} e^x (2\cos 2x + 2\sin 2x)$$

$$= -\frac{1}{8} e^x (\cos 2x + \sin 2x).$$

Find the P.I. of $(D^2 + 5D + 4)y = x^2 + 7x + 9$.

Sol.

$$\text{P.I.} = \frac{1}{D^2 + 5D + 4}(x^2 + 7x + 9) = \frac{1}{4\left(1 + \frac{5D}{4} + \frac{D^2}{4}\right)}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 + \left(\frac{5D}{4} + \frac{D^2}{4}\right)\right]^{-1}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 - \left(\frac{5D}{4} + \frac{D^2}{4}\right) + \left(\frac{5D}{4} + \frac{D^2}{4}\right)^2 - \ldots\ldots\right](x^2 + 7x + 9)$$

$$= \frac{1}{4}\left(1 - \frac{5D}{4} - \frac{D^2}{4} + \frac{25D^2}{16} \ldots\ldots\right)(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left(1 - \frac{5D}{4} + \frac{21D^2}{16} \ldots\ldots\right)(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[(x^2 + 7x + 9) - \frac{5}{4}D(x^2 + 7x + 9) + \frac{21}{16}D^2(x^2 + 7x + 9)\right]$$

$$= \frac{1}{4}\left[(x^2 + 7x + 9) - \frac{5}{4}(2x + 7) + \frac{21}{16}(2)\right] = \frac{1}{4}\left(x^2 + \frac{9}{2}x + \frac{23}{8}\right).$$

Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$. Given $y(0) = 0,\ y'(0) = 1$.

Sol. Given differential equation can be written as

$$(D^2 + 4D + 5)y = -2\cosh x$$

A.E. is $D^2 + 4D + 5 = 0 \quad \Rightarrow \quad D = -2 \pm i$

$$\text{C.F. } y = e^{-2x}(c_1 \cos x + c_2 \sin x)$$

$$\text{P.I. } y = \frac{1}{D^2 + 4D + 5}(-2\cosh x)$$

$$= \frac{-1}{D^2 + 4D + 5}(e^x + e^{-x})$$

$$= \frac{-1}{D^2 + 4D + 5} e^x + \frac{-1}{D^2 + 4D + 5} . e^{-x}$$

$$= \frac{-1}{10} e^x + \frac{-1}{2} e^{-x} = -\frac{1}{10}(e^x + 5e^{-x})$$

Using (1) and (2), the complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^{-2x}(c_1 \cos x + c_2 \sin x) - \frac{1}{10}(e^x + 5e^{-x})$$

Given $y(0) = 0 \quad \Rightarrow \quad 0 = c_1 - \frac{1}{10} - \frac{1}{2} \quad \Rightarrow \quad c_1 = \frac{3}{5},$

Solve the differential equation $(D^2 + 2D - 3)y = x^2 e^{-3x}$.

Sol. Given differential equation is $(D^2 + 2D - 3)y = x^2 e^{-3x}$.

A.E. is $\quad D^2 + 2D - 3 = 0 \Rightarrow (D - 1)(D + 3) = 0$

$\Rightarrow \quad D = 1, -3$

These are real and unequal.

So C.F. is $\quad y = c_1 e^{x} + c_2 e^{-3x}$

$$\text{P.I. } y = \frac{1}{D^2 + 2D - 3} \cdot x^2 e^{-3x} = e^{-3x}\left[\frac{1}{(D-3)^2 + 2(D-3) - 3}\right]x^2$$

$$= e^{-3x}\left[\frac{1}{D^2 - 4D}\right]x^2 = e^{-3x}\frac{1}{-4D\left(1 - \frac{D^2}{4D}\right)}x^2$$

$$= e^{-3x}\left\{\frac{-1}{4D}\left[1 + \frac{D}{4} + \frac{D^2}{16} + \ldots\right]x^2\right\}$$

$$= e^{-3x}\left\{\frac{-1}{4D}\left[x^2 + \frac{1}{4}.2x + \frac{1}{16}.2 + 0\right]\right\}$$

$$= e^{-3x} - \frac{1}{4}\left[\frac{x^3}{3} + \frac{1}{2}.\frac{x^2}{2} + \frac{1}{8}.x\right]$$

Solve $(D^2 + 1)\, y = \sin x \sin 2x + e^x x^2$.

Sol. Given differential equation is $(D^2 + 1) = \sin x \sin 2x + e^x . x^2$

A.E. is $\quad D^2 + 1 = 0 \;\Rightarrow\; D = \pm i$

$$\text{C.F. } y = c_1 \cos x + c_2 \sin x. \qquad \text{...(1)}$$

$$\text{P.I.} = \frac{1}{D^2+1}(\sin x \sin 2x + e^x x^2) = \frac{1}{D^2+1} \sin x \sin 2x + \frac{1}{D^2+1} x^2 . e^x$$

$$\text{Consider } \frac{1}{D^2+1} \sin x \sin 2x = \frac{1}{D^2+1}\left(\frac{\cos x - \cos 3x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2+1}\cos x - \frac{1}{D^2+1}\cos 3x\right] = \frac{1}{2}\left[x \sin x + \frac{1}{8}\cos 3x\right]$$

$$\text{Next, } \frac{1}{D^2+1} x^2 e^x = e^x \frac{1}{(D+1)^2+1} x^2 = e^x \frac{1}{D^2+2D+2} x^2$$

$$= e^x \frac{1}{2}\left(\frac{1}{1+\frac{D^2+2D}{2}} x^2\right) = \frac{e^x}{2}\left[1 - \frac{D^2+2D}{2} + \left(\frac{D^2+2D}{2}\right)^2 - \ldots\ldots\right] x^2$$

$$= \frac{e^x}{2}\left[x^2 - \left(\frac{1+2x}{2}\right) + \frac{1}{4}(8)\right] = \frac{e^x}{2}\left[x^2 - 2x + \frac{3}{2}\right]$$

$$\text{So} \qquad \text{P.I. } y = \frac{x \sin x}{2} + \frac{\cos 3x}{16} + \frac{e^x}{2}\left(x^2 - x + \frac{3}{2}\right) \qquad \text{...(2)}$$

Using (1) and (2), the complete solution is